# 5. Graphs and matrices

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Similarly, letting $\mathfrak{F}_{x,e,y}$ count the paths from $x$ to $y$ that start with the edge $e$,

$$\begin{aligned}\mathfrak{F}_{x,y} &= \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \mathfrak{F}_{x,e,y}\,,\\ \mathfrak{F}_{x,e,y} &= e^\ell\left(\mathfrak{F}_{e^\omega,y} + (u-1)\mathfrak{F}_{e^\omega,\overline{e},y}\right)\,,\\ \mathfrak{F}_{e^\omega,\overline{e},y} &= \overline{e}^\ell\left(\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y}\right)\,;\end{aligned}$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = \left(1-(u-1)^2(e\overline{e})^\ell\right)^{-1}\left(e^\ell\mathfrak{F}_{e^\omega,y} + (u-1)(e\overline{e})^\ell\mathfrak{F}_{x,y}\right)\,,$$

which we insert in the first line to obtain

$$K_x^{-1}\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\cdot K_{e^\omega}^{-1}\mathfrak{F}_{e^\omega,y}\,.$$

Thus if we let

$$e^{\ell'} = \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\,,\qquad A' = \sum_{e\in E(\mathcal{X})}[e^{\ell'}]_{e^\alpha}^{e^\omega}\,, \tag{4.1}$$

we obtain

$$(K_x^{-1}\mathfrak{F}_{x,y})_{x,y\in V(\mathcal{X})} = \frac{1}{1-A'} \tag{4.2}$$

and the proof is finished in the case that $\mathcal{X}$ is finite, because the matrix $A'$ is precisely that obtained from $A$ by substituting $\ell'$ for $\ell$.

If $\mathcal{X}$ has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by $\mathfrak{F}^n_{\star,\dagger}(\ell)$ and $\mathfrak{G}^n_{\star,\dagger}(\ell')$ the enriched path series and path series respectively in $\mathcal{B}(\star,n)$, and write

$$K_\star\cdot\mathfrak{F}(\ell) = \lim_{n\to\infty}\mathfrak{F}^n_{\star,\dagger}(\ell) = \lim_{n\to\infty}\mathfrak{G}^n_{\star,\dagger}(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

## 5. Graphs and matrices

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand '$X/Y$' as '$X\cdot Y^{-1}$'.

DEFINITION 5.1. Let $\mathcal{X}$ be a finite graph. The *edge-adjacency* and *inversion* matrices of $\mathcal{X}$, respectively $B$ and $J$, are $E(\mathcal{X}) \times E(\mathcal{X})$ matrices over $\mathbf{Z}$ defined by

$$B_{e,f} = \begin{cases} 1 & \text{if } e^\omega = f^\alpha \\ 0 & \text{else,} \end{cases} \qquad J_{e,f} = \begin{cases} 1 & \text{if } \overline{e} = f \\ 0 & \text{else.} \end{cases}$$

The *vertex-adjacency* and *degree* matrices of $\mathcal{X}$, respectively $A$ and $D$, are $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{Z}$ defined by

$$A_{v,w} = |\{e \in E(\mathcal{X}) \mid e^\alpha = v \text{ and } e^\omega = w\}|, \quad D_{v,w} = \begin{cases} \deg(v) & \text{if } v = w, \\ 0 & \text{else.} \end{cases}$$

A *cycle* is the equivalence class of a circuit under cyclic permutation of its edges. A *proper cycle* is a cycle all of whose representatives are proper circuits. A cycle is *primitive* if none of its representatives can be written as $\pi^k$ for some $k \geq 2$. The *cyclic bump count* $\mathrm{cbc}(\pi)$ of a circuit $\pi = (\pi_1, \ldots, \pi_n)$ is

$$\mathrm{cbc}(\pi) = |\{i = 1, \ldots, n \mid \pi_i = \overline{\pi_{i+1}}\}| \,,$$

where the edge $\pi_{n+1}$ is understood to be $\pi_1$.

The matrices given above are related to paths in $\mathcal{X}$ as follows: Consider first the matrix

$$M = \mathbf{1} - (B - (1-u)J)t\,.$$

Then the $(e,f)$ coefficient of $M^{-1}$ is precisely

$$\sum_{\pi : \pi_1 = e, \pi^\omega = f^\alpha} u^{\mathrm{bc}(\pi f)} t^{|\pi|}\,.$$

This is clear because the series expansion of $M^{-1}$ is the sum of sequences of $(B-J)t$ (contributing edges with no bump) and $Jut$ (contributing edges with bumps), with an extra factor of $u$ in case the path ends in $\overline{f}$. As a consequence,

LEMMA 5.2. *Let*

$$X_E = \frac{\mathbf{1} + (1-u)Jt - M}{Mt} = \frac{B}{\mathbf{1} - (B - (1-u)J)t}\,.$$

*Then the $(e,f)$ coefficient of $X_E$ counts the non-trivial paths starting with $e$ and ending at $f^\alpha$, with $t$-weight shifted one down:*

$$(X_E)_{e,f} = \sum_{\pi : \pi_1 = e,\ \pi^\omega = f^\alpha} u^{\mathrm{bc}(\pi)} t^{|\pi|-1}\,.$$

Likewise, consider the matrix

$$P = \mathbf{1} - At + (1-u)(D - (1-u)\mathbf{1})t^2 .$$

The following lemma will be a consequence of the computations in the next section.

LEMMA 5.3. *Let*

$$X_V = \frac{(1-(1-u)^2t^2)\mathbf{1} - P}{Pt} = \frac{A - (1-u)Dt}{\mathbf{1} - At + (1-u)(D-(1-u)\mathbf{1})t^2} .$$

*Then the* $(v,w)$ *coefficient of* $X_V$ *counts the non-trivial paths starting at* $v$ *and ending at* $w$*, with* $t$*-weight shifted one down:*

$$(X_V)_{v,w} = \sum_{\pi \,:\, \pi^\alpha = v,\ \pi^\omega = w} u^{\mathrm{bc}(\pi)} t^{|\pi|-1} .$$

*Proof.* We will show the matrix $\mathbf{1} + X_V t$ has as $(v,w)$ coefficient the enriched path series from $v$ to $w$. By simple calculation

$$\mathbf{1} + X_V t = \frac{\mathbf{1} - (1-u)^2t^2}{\mathbf{1} - At + (1-u)(D-(1-u)\mathbf{1})t^2} = \frac{K^{-1}}{\mathbf{1} - A'} ,$$

where $K$ and $A'$ are given by

$$K = \frac{\mathbf{1} + (1-u)(D - 1 + u)t^2}{1-(1-u)^2t^2}, \qquad A' = \frac{AKt}{1-(1-u)^2t^2} .$$

$K$ is a diagonal matrix and the coefficient $K_{x,x}$ is precisely $K_x$ for the length labelling, while the matrix $A'$ is the matrix of (4.1) in the previous section. The result then follows from Equation (4.2). □

In particular, the two matrices $X_E$ and $X_V$ have the same trace, as this trace counts all the non-trivial circuits $\pi$ in $\mathcal{X}$, with weight $u^{\mathrm{bc}(\pi)} t^{|\pi|-1}$.

We now state and prove an extension of a theorem by Bass [Bas92, FZ98, Nor96]:

THEOREM 5.4. *Let* $C$ *be a set of representatives of primitive cycles in* $\mathcal{X}$*, and form the zeta function of* $\mathcal{X}$

$$\zeta(u,t) = \prod_{\gamma \in C} \frac{1}{1 - u^{\mathrm{cbc}(\gamma)} t^{|\gamma|}} .$$

*(The choice of representatives does not change the zeta function.) Then* $\zeta^{-1}$ *is a polynomial in* $u$ *and* $t$ *and can be expressed as*

$$\frac{1}{\zeta(u,t)} = \det M \tag{5.1}$$

$$= (1+(1-u)t)^n(1-(1-u)^2t^2)^{m-|V(\mathcal{X})|}\det P\,, \tag{5.2}$$

*where*

$$n = |\{e \in E(\mathcal{X}) \mid e = \overline{e}\}|\,, \qquad 2m = |\{e \in E(\mathcal{X}) \mid e \neq \overline{e}\}|\,.$$

The special case $u = n = 0$ of this result was stated and proved in the given sources. We will prove the general statement, using a result of Shimson Amitsur:

THEOREM 5.5 (Amitsur [Ami80,RS87]). *Let* $X_1,\dots,X_k$ *be square matrices of the same dimension over an arbitrary ring. Let* $S$ *contain one representative up to cyclic permutation of words over the alphabet* $\{1,\dots,k\}$ *that are primitive, i.e. such that none of their cyclic permutations are proper powers of a word (*$S$ *is infinite as soon as* $k > 1$*). For* $p = i_1\dots i_n \in S$ *set* $X_p = X_{i_1}\dots X_{i_n}$*. Then*

$$\det(\mathbf{1}-(X_1+\cdots+X_k)t) = \prod_{p\in S}\det(\mathbf{1}-X_pt^{|p|})\,,$$

*considered as an equality of power series in* $t$ *over the matrix ring.*

The equality (5.1) then follows; indeed, for all edges $e \in E(\mathcal{X})$ let $X_e$ be the $E(\mathcal{X}) \times E(\mathcal{X})$ matrix whose $e$-th row is the $e$-th row of $B-(1-u)J$, and whose other rows are $0$. Then clearly $\mathbf{1}-\sum_{e\in E(\mathcal{X})}X_et = M$ and, for any sequence of edges $\pi$,

$$\det(\mathbf{1}-X_\pi t^{|\pi|}) = \begin{cases} 1-u^{\mathrm{cbc}(\pi)}t^{|\pi|} & \text{if } \pi \text{ is a circuit,}\\ 1 & \text{else,}\end{cases}$$

so equality of $\zeta(u,t)$ and $\det M$ follows from Amitsur's Theorem.

To prove (5.2), we use the following result, whose proof relies on Newton's formulas relating the trace of powers of $X$ and the characteristic polynomial of $X$:

PROPOSITION 5.6 ([Ami80, Equation 4.4)]. *Let $X$ be a power series in $t$ over a matrix ring, such that $X(0) = \mathbf{1}$. Then*

$$\det X = \exp\Big(-\int \operatorname{tr}\Big(\frac{\mathbf{1}-X}{Xt}\Big)dt\Big) ,$$

*where the integration is the formal linear operation on power series that maps $t^r$ to $t^{r+1}/(r+1)$.*

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned}
\frac{\det M}{(1+(1-u)t)^n(1-(1-u)^2t^2)^m} &= \det \frac{M}{\mathbf{1}+(1-u)Jt}\\
&= \exp\Big(-\int \operatorname{tr}\frac{\mathbf{1}+(1-u)Jt-M}{Mt}dt\Big)\\
&= \exp\Big(-\int \begin{matrix}\text{series counting non-trivial circuits,}\\ \text{length shifted down by one}\end{matrix}\; dt\Big)\\
&= \exp\Big(-\int \operatorname{tr}\frac{(1-(1-u)^2t^2)\mathbf{1}-P}{Pt}dt\Big)\\
&= \det\frac{P}{1-(1-u)^2t^2} = \frac{\det P}{(1-(1-u)^2t^2)^{|V(\mathcal{X})|}} .
\end{aligned}$$

## 6. SECOND PROOF OF THEOREM 2.4

Let $P = [\star, \dagger]$ be the set of paths in $\mathcal{X}$ from $\star$ to $\dagger$. As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in $\Pi = \mathbf{Z}[[P]]$, the $\mathbf{Z}$-module of functions from the set of paths to $\mathbf{Z}$. We embed subsets of $P$ in $\Pi$ by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A , \quad \text{with } (\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A ,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathcal{B}$ be the subset of bounded non-negative elements of $\Pi$ (i.e. functions $f$ such that there is a constant $N$ with $0 \le (\pi)f < N$ for all paths $\pi$). If $\ell$ is a complete labelling of $\mathcal{X}$, there is an induced labelling $\ell_*\colon \mathcal{B} \to \mathbf{k}$ given by

$$(f)\ell_* = \sum_{\pi\in P} (\pi) f \pi^\ell .$$

Note that the sum, although infinite, defines an element of $\mathbf{k}$ due to the fact that $\ell$ is complete.