5. Parallel transport in a cubical manifold and the proof of theorem 3

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the sequence of curves $r \circ \gamma_m$ converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics γ_m converges locally uniformly. By definition, this means that the corresponding subsequence of (x_m) converges to a point $\xi \in X(\infty)$.

Let $\phi \in \Gamma$ and choose $c = c_\phi$ as in Lemma 4.8. Let $t_0 > 0$ be given. By Lemma 4.8 we have $r \circ \gamma_m(t_0) \in F$ for all $m \geq m_0$. By Proposition 3.4 and Lemma 4.8, we have $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{n} c_\phi$ for all such m. Now c_ϕ is independent of t_0, hence $\phi(\xi) = \xi$. □

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1, $\Delta \cong \ker h$ consists precisely of the elliptic elements of Γ. If indices of type 1 do not occur, then Proposition 4.3 applies: If $k = 0$, then $\Gamma \cong \Delta$ fixes a point of X and possibility (1) holds. If $k > 0$, then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that $\mathrm{Stab}(x) \neq \Delta$ for any $x \in X$ in this case since Δ would have a fixed point otherwise.

5. Parallel transport in a cubical manifold and the proof of Theorem 3

Let X be a cubical manifold of dimension n. Given two chambers P and Q in X with a common face of dimension $n-1$, we define $t_{PQ} \colon P \to Q$ to be the *translation* which moves each point p of P along the unit geodesic segment starting at p and orthogonal to the common $(n-1)$-face of P to the end point in Q. The map t_{PQ} is an isomorphism and isometry of P with Q. Given a gallery $\pi = (P_1, \dots, P_n)$ in X, the *parallel transport* along π is the isomorphism $t_\pi \colon P_1 \to P_n$ given by

$$t_\pi := t_{P_{n-1}P_n} \circ \cdots \circ t_{P_2P_3} \circ t_{P_1P_2} .$$

LEMMA 5.1. *Let X be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. Then for any two chambers P and Q in X, the parallel transport t_π along a gallery π connecting P and Q is independent of π.*

Proof. It is enough to show that the parallel transport along any closed gallery is the identity. Let π be such a gallery with initial and final chamber P.

Represent π by a closed curve c which starts and ends in some interior point p of P, such that c misses the $(n-2)$-skeleton of X and crosses $(n-1)$-faces transversally and according to the pattern provided by π. Since X is simply connected, the curve c can be contracted in X to the point p. Since X is a manifold, the links of the vertices in X are $(n-2)$-connected. Hence the contraction of c can be chosen to be generic in the sense that it misses the $(n-3)$-skeleton of X and crosses the $(n-2)$-skeleton transversally. Following the curve c along this contraction, we get a sequence of modifications of the gallery π. These modifications occur when c crosses an $(n-2)$-face of X. The condition that the number of chambers adjacent to such faces is divisible by 4 implies that the parallel transport t_π does not change under these modifications. Since the parallel transport along the trivial gallery is the identity, $t_\pi = \mathrm{id}_P$. $\square$

From now on we assume that X is a simply connected cubical manifold such that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. For chambers P and Q in X define $t_{PQ} = t_\pi$, where π is any gallery connecting P with Q. The above lemma shows that t_{PQ} is well defined.

We fix a chamber P_0 of X and define a homomorphism $\phi\colon \Gamma \to \operatorname{Aut} P_0$ by

$$\phi(g) := t_{g(P_0)P_0} \circ g|_{P_0}\,.$$

The kernel $\Gamma' := \ker\phi$ is a finite index subgroup of Γ and consists precisely of those automorphisms of Γ whose restriction to any chamber commutes with the corresponding parallel transport.

COORIENTATIONS

A *coorientation of a wall* in a chamber is a choice of one of the two half-chambers determined by the wall. Once and for all, we choose coorientations of the walls in the above chamber P_0. Now by Lemma 5.1, parallel transport gives rise to a consistent choice of coorientations for all walls in X.

By Corollary 1.3, X is foldable. We fix a folding and denote by Λ_i the set of hyperspaces of X with label i. Note that Λ_i is invariant under parallel transport. Along a hyperspace with label i, the half-chambers distinguished by the coorientation are all contained in the same halfspace with respect to the hyperspace. The above group Γ' preserves the families Λ_i together with the coorientations.

The index of intersection of an oriented curve c at a transversal crossing of a hyperspace $H \in \Lambda_i$ is defined to be equal to $+1$ or -1 respectively, according to whether the orientation of c coincides with the coorientation of H or not. Fix a point p_0 in the interior of P_0 which does not belong to any wall and any of the chosen coorientations. For $p \in X$ define $f_i(p)$ to be the sum of the indices of intersection of an oriented curve c connecting p_0 and p with the hyperspaces from Λ_i. Here we assume that c is generic, i.e. c does not meet the $(n-2)$-skeleton and crosses hyperspaces transversally. The integer $f_i(p)$ does not depend on c since X is simply connected and any two such curves can be deformed into each other by a homotopy which misses the $(n-3)$-skeleton of X and crosses the $(n-2)$-skeleton of X transversally.

For $g \in \Gamma$ set $h_i(g) = f_i(g(p_0))$. Since the chosen system of coorientations is invariant under the action of Γ', the maps $h_i \colon \Gamma' \to \mathbf{Z}$ are homomorphisms. We finish the proof of Theorem 3 by showing that the image of $h = (h_1, \ldots, h_n)$ is of finite index in $\mathbf{Z}^n$.

We need to show that the image of h contains n linearly independent vectors. To that end, we show that the image contains non-zero vectors which span arbitrarily small angles with the unit vectors e_i in $\mathbf{R}^n$, $1 \leq i \leq n$. Let σ be a unit speed geodesic ray with $\sigma(0) = p_0$ which is perpendicular in P_0 to the wall with label i. By the choice of p_0, the ray σ does not meet the $(n-2)$-skeleton of X and is perpendicular to all $(n-1)$-faces and walls with label i which it intersects. We have

$$f_j(\sigma(m)) = \delta_{ij} \cdot m, \quad m \geq 1 .$$

By the cocompactness of the action of Γ', there is an integer $k \geq 1$ such that for any $m \geq 1$ there is a $g_m \in \Gamma'$ with $d(\sigma(m), g_m(p_0)) \leq k$. By the definition of h_j this implies $|h_j(g_m) - f_j(\sigma(m))| \leq k$. Theorem 3 follows.