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A SHORT PROOF OF A THEOREM OF CAMACHO AND SAD

by Matei TOMA

In [2] Camacho and Sad answer a long standing open question by proving the following

THEOREM. *Any holomorphic vector field defined in a neighborhood of the origin in* $\mathbf{C}^2$ *admits an invariant complex curve passing through* $0 \in \mathbf{C}^2$.

The interesting case is of course when the vector field has an isolated zero at the origin. It then defines a singular holomorphic foliation around $0 \in \mathbf{C}^2$ and the required invariant curve C (which we assume irreducible) has the property that $C \setminus \{0\}$ is a leaf of this foliation. Alternatively, one may consider a holomorphic 1-form ω inducing the same foliation. Then for C to be invariant means $i^*\omega \equiv 0$ where i is the inclusion map of C into some neighborhood of $0 \in \mathbf{C}^2$.

The proof given in [2] relies on a reduction theorem for the singularities of a vector field and on the introduction of an index of a foliation at a singular point with respect to an invariant curve.

In this note we shall combine these ingredients with the negativity of the intersection form (which we introduce in an elementary way) on the components of the exceptional divisor of a sequence of blow-ups to give a short proof of the above theorem. (As the referee points out, a different use of the intersection form is made in [1] in order to generalize the same theorem.)

We begin by recalling the reduction theorem and the definition of the Camacho-Sad index and refer the reader to [2] for more details on these points.

Let U be an open neighborhood of $0 \in \mathbf{C}^2$ and suppose that there is a holomorphic 1-form ω on U vanishing only at $0 \in \mathbf{C}^2$ and defining our singular foliation $\mathcal{F}$. Further let $\widetilde{U}$ be the complex surface obtained by blowing up $0 \in U$ and $\pi\colon \widetilde{U} \to U$ the projection. The 1-form $\pi^*\omega$ vanishes on the exceptional divisor $E \cong \mathbf{P}^1_{\mathbf{C}}$ but after dividing $\pi^*\omega$ locally by a sufficiently high power of the defining function of E we obtain a 1-form with isolated zeros only. This induces a foliation $\widetilde{\mathcal{F}}$ on $\widetilde{U}$. It is clear that $\widetilde{\mathcal{F}}|_{\widetilde{U}\setminus E}$ is the same as $\mathcal{F}|_{U\setminus\{0\}}$ and that $\widetilde{\mathcal{F}}$ has finitely many singularities all located on E.

We may assume that E is an invariant curve of $\widetilde{\mathcal{F}}$. Otherwise the projection of an invariant curve of $\widetilde{\mathcal{F}}$ passing through a non-singular point on E would be a solution to our problem.

We recall the following:

DEFINITION. With respect to local coordinates (x, y) around 0 one writes $\omega(x,y) = -B(x,y)\,dx + A(x,y)\,dy$. Then we say that the point $0 \in U$ is an *irreducible singularity* for $\mathcal{F}$ if the associated variational matrix at $0 \in U$

$$\begin{pmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \end{pmatrix}(0)$$

has eigenvalues λ_1, λ_2 with either

$$(*) \qquad \lambda_1 \neq 0,\ \lambda_2 \neq 0 \text{ and } \frac{\lambda_1}{\lambda_2},\ \frac{\lambda_2}{\lambda_1} \notin \mathbf{N}$$

or

$$(**) \qquad \lambda_1 \neq 0,\ \lambda_2 = 0 \text{ or } \lambda_1 = 0,\ \lambda_2 \neq 0\,.$$

(The definition is independent of the local coordinates used. In fact λ_1, λ_2 are up to a factor invariants of the foliation.)

For irreducible singularities there exist normal forms due to Dulac (cf. [4]). In particular, this means that one can always find a holomorphic change of coordinates such that in case $(*)$

$$\omega(x,y) = -yh(x,y)\,dx + xg(x,y)\,dy$$

and in case $(**)$ when $\lambda_1 \neq 0$ and $\lambda_2 = 0$

$$\omega(x,y) = -yh(x,y)\,dx + (\lambda_1 x + yg(x,y))\,dy$$

for suitable holomorphic functions g and h.

Hence one sees that in case $(*)$ $\mathcal{F}$ admits two smooth invariant curves $C_1 = \{y = 0\}$, $C_2 = \{x = 0\}$ tangent to the eigenspaces associated to λ_1, λ_2 respectively, and in case $(**)$ one invariant curve $C_1 = \{y = 0\}$ tangent to the eigenspace associated to $\lambda_1 \neq 0$. In case $(**)$ $\mathcal{F}$ admits a second formal invariant curve (associated to λ_2) which is in general not convergent.

REDUCTION THEOREM (cf. [4]). *After a finite number of blow-ups all singularities of the induced foliation are irreducible.*

DEFINITION. Let C be a smooth invariant curve of a foliation $\mathcal{F}$ on U and $0 \in U \cap C$. Suppose that local coordinates are chosen such that $C = \{(x,y) \in U \mid y = 0\}$ and let $\omega(x,y) = -B(x,y)\,dx + A(x,y)\,dy$ be a holomorphic 1-form which induces $\mathcal{F}$. Then the *Camacho-Sad index* of $\mathcal{F}$ at $0 \in U$ with respect to C is defined by

$$CS(\mathcal{F}, C, 0) := \mathrm{Res}_{x=0} \frac{\partial}{\partial y}\left(\frac{B}{A}\right)\Big|_C .$$

It is not difficult to prove that the definition doesn't depend on the choice of ω and local coordinates (cf. [3] for a generalization).

Notice that the index vanishes if 0 is non-singular for $\mathcal{F}$. Using the normal forms for irreducible singularities one sees immediately that

$$CS(\mathcal{F}, C_1, 0) = \frac{\lambda_2}{\lambda_1} \neq 0, \quad CS(\mathcal{F}, C_2, 0) = \frac{\lambda_1}{\lambda_2} \neq 0 \text{ in case } (*)$$

and

$$CS(\mathcal{F}, C_1, 0) = 0 \text{ in case } (**) \text{ when } \lambda_1 \neq 0,\ \lambda_2 = 0 .$$

The main property of the Camacho-Sad index is that in some sense it localizes the self-intersection of a compact invariant curve at the singular points of the foliation ([2], Appendix). We choose here to remain elementary so we shall use this property only for the components of the exceptional divisor appearing after a sequence of blow-ups of smooth points. In this case this will be a consequence of the following two easy lemmata which are proven in [2]:

LEMMA 1. *If the exceptional curve E of the blow-up of a singular point of a foliation $\mathcal{F}$ is invariant for the induced foliation $\widetilde{\mathcal{F}}$ then*

$$\sum_{P \in E} CS(\widetilde{\mathcal{F}}, E, P) = -1 .$$

LEMMA 2. *Let C be a smooth invariant curve for $\mathcal{F}$ passing through $0 \in U$, $\widetilde{C}$ the strict transform of C after blowing up 0, $\widetilde{\mathcal{F}}$ the induced foliation and E the exceptional curve of this blow-up. Then*

$$CS(\widetilde{\mathcal{F}}, \widetilde{C}, \widetilde{C} \cap E) = CS(\mathcal{F}, C, 0) - 1 .$$

Consider now the components $C_1, \ldots, C_n$ of the exceptional divisor appearing after a sequence of n blow-ups of an isolated singularity of a holomorphic foliation in dimension 2 and suppose that each C_i is invariant for the induced foliation $\widetilde{\mathcal{F}}$. We define then the following "intersection" matrix:

$$a_{ij} = \begin{cases} 1 & \text{when } i \neq j \text{ and } C_i \cap C_j \neq \varnothing\,; \\ 0 & \text{when } i \neq j \text{ and } C_i \cap C_j = \varnothing\,; \\ \sum_{P \in C_i} CS(\widetilde{\mathcal{F}}, C_i, P) & \text{when } i = j\,. \end{cases}$$

One checks immediately by induction on n and using the lemmata that $(a_{ij})_{1 \leq i,j \leq n}$ is integer valued and negative definite. Indeed, suppose that a blow-up occurs at a point P of C_n and let $x' \in \mathbf{R}^n, x = (x', x_{n+1}) \in \mathbf{R}^{n+1}$. If we denote by $\langle x', x' \rangle := \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} a_{ij} x_i x_j$ the old intersection form and by $\langle x, x \rangle$ the new one, we get

$$\langle x, x \rangle = \langle x', x' \rangle - x_n^2 + 2x_n x_{n+1} - x_{n+1}^2 = \langle x', x' \rangle - (x_n - x_{n+1})^2$$

in case P didn't belong to another curve C_i, $1 \leq i \leq n-1$ and

$$\begin{aligned} \langle x, x \rangle &= \langle x', x' \rangle - x_n^2 - x_{n-1}^2 + 2x_n x_{n+1} + 2x_{n-1} x_{n+1} - 2x_n x_{n-1} - x_{n+1}^2 \\ &= \langle x', x' \rangle - (x_n + x_{n-1} - x_{n+1})^2 \end{aligned}$$

in case $P = C_n \cap C_{n-1}$.

We can now give the

Proof of the theorem. Let $\mathcal{F}$ be a holomorphic foliation having an isolated singularity at the origin of $\mathbf{C}^2$. Suppose that there is no invariant curve passing through $0 \in \mathbf{C}^2$.

By the Reduction Theorem after finitely many blow-ups, say n, all the singularities of the induced foliation $\widetilde{\mathcal{F}}$ on the exceptional divisor $D = C_1 + \cdots + C_n$ will be irreducible. The components $C_1, \ldots, C_n$ have to be invariant for $\widetilde{\mathcal{F}}$. Their intersection points $C_i \cap C_j$ are singularities of $\widetilde{\mathcal{F}}$ which are either of type $(*)$ or of type $(**)$ with two invariant curves. Since an invariant curve for $\widetilde{\mathcal{F}}$ transverse to the exceptional divisor would give by projection an invariant curve for $\mathcal{F}$ through 0, any singularity P lying on some C_i and not an intersection point $C_i \cap C_j$ has to be of type $(**)$ with C_i as unique invariant curve (associated to the non-zero eigenvalue). Thus for such a singularity P one has $CS(\widetilde{\mathcal{F}}, C_i, P) = 0$.

Take now an intersection point $P_1 = C_{i_1} \cap C_{j_1}$, $i_1 \neq j_1$ which is a singularity of type $(**)$ and suppose that C_{i_1} is tangent to the eigenspace belonging to the non-zero eigenvalue of the variational matrix at P_1. Then P_1 splits the exceptional divisor into two connected parts, $D = D_{i_1} + D_{j_1}$ such that D_{i_1} contains C_{i_1} and D_{j_1} contains C_{j_1}. We retain D_{i_1} as interesting for our argument. If P_2 is an intersection point $C_{i_2} \cap C_{j_2}$ on D_{i_1} of type $(**)$ we split D_{i_1} into $D_{i_2} + D_{j_2}$ and so on. We end up with a connected part D_{i_k} of D such that all intersection points of components of D_{i_k} are of type $(*)$ and for any other singularity P of $\widetilde{\mathcal{F}}$ lying on an irreducible component C_j of D_{i_k} one has

$$CS(\widetilde{\mathcal{F}}, C_j, P) = 0\,.$$

By renumbering we can assume that

$$D_{i_k} = C_1 + \cdots + C_m,\ 1 \leq m \leq n\,.$$

We construct now an eigenvector $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{C}^m$ of $(a_{ij})_{1\leq i,j\leq m}$ with $\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m \neq 0$. Set $\alpha_1 = 1$ and if $C_1 \cap C_i \neq \varnothing$ for $2 \leq i \leq n$ let

$$\alpha_i := -\alpha_1 CS(\widetilde{\mathcal{F}}, C_1, C_1 \cup C_i)\,.$$

Continue by putting

$$\alpha_j := -\alpha_i CS(\widetilde{\mathcal{F}}, C_i, C_i \cup C_j)$$

for $C_j \cap C_i \neq \varnothing$ and $j \neq i$, $1 \leq j \leq n$, etc.

Since D_{i_k} is connected and contains no cycles we get the desired vector α. The scalar product of α with the j-th column of $(a_{ij})_{1\leq i,j\leq m}$ is

$$\begin{aligned}\sum_{i=1}^{m} \alpha_i \cdot a_{ij} &= \alpha_j a_{jj} + \sum_{\substack{1\leq i\leq m\\ i\neq j}} \alpha_i a_{ij}\\
&= \alpha_j \Bigg(\sum_{P\in C_j} CS(\widetilde{\mathcal{F}}, C_j, P) + \sum_{\substack{1\leq i\leq m\\ i\neq j\\ C_i\cap C_j\neq\varnothing}} \frac{\alpha_i}{\alpha_j} \Bigg)\\
&= \alpha_j \Bigg(\sum_{P\in C_j} CS(\widetilde{\mathcal{F}}, C_j, P) - \sum_{\substack{1\leq i\leq m\\ i\neq j\\ C_i\cap C_j\neq\varnothing}} CS(\widetilde{\mathcal{F}}, C_j, C_i \cap C_j) \Bigg)\\
&= 0\end{aligned}$$

so 0 is an eigenvalue of $(a_{ij})_{1\leq i,j\leq m}$.

But $(a_{ij})_{1\leq i,j\leq m}$ is negative definite since $(a_{ij})_{1\leq i,j\leq n}$ was, hence a contradiction. □

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