5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

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immersions. Given $v \in V'$, we need to show there exists $x \in V'$ such that (x, v) and (v, x) are in Z'. This is true if $v \in V$ by the property of Z. If $v \in V_s$, then $v = a_s$ for some $a \in V$. We leave it to the reader to show that $(x, a_s) \in Z_1$ and $(a_s, x) \in Z_2$ for generic x in V. This completes the proof of the lemma.

The above lemma allows us to replace V by V', hence to expand V whenever there exists a point s in V such that vs is not defined for all $v \in V$, and we can expand V' if there exists a point $s' \in V'$ such that $v's'$ is not defined for all $v' \in V'$. Denote the result of finitely many such expansions also by V', and let $U \subset V \times V \times V'$ be the closure of Γ. By Lemma 4.3 applied to V', the projection $p_{12}\colon U \to V \times V$ is an open immersion. Its image is the set of points (a, b) such that $m\colon V \times V \to V'$ is defined at (a, b). If $V \times s \not\subset p_{12}(U)$ for some point s in V, then replacing V' by $V' \cup V_s'$ increases both V' and $p_{12}(U)$. Using noetherian induction on open subschemes of $V \times V$, we may assume that after finitely many expansions, $V \times s \subset p_{12}(U)$ for all points $s \in V$. Then we have $p_{12}(U) = V \times V$.

PROPOSITION 4.5. *Let V, V', and U be as above. If $p_{12}(U) = V \times V$, then the operation $m\colon V' \times V' \to V'$ is everywhere defined on V' and makes V' an algebraic group.*

Proof. Take (a', b') in $V' \times V'$. Choose a point x so that $a'x$ and $x^{-1}b'$ are both defined and lie in V. Then we can define $m(a', b') = (a'x)(x^{-1}b')$. Similarly one can define $a'^{-1}b'$ and $b'a'^{-1}$. In this way we extend m, Φ, Ψ, Φ^{-1} and Ψ^{-1} to $V' \times V'$. The verification of the group axioms is routine and is omitted.

5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

Keep the notations in §3. We have proved that there is a birational group structure on $(X-S)^{(\pi)}$. The algebraic group associated to this birational group is called the *generalized jacobian* of $X_{\mathfrak{m}}$ and is denoted by $J_{\mathfrak{m}}$. It is a commutative algebraic group.

Let D_0 be a divisor on X prime to S of degree 0. By Lemma 3.3, the set

$$V_{D_0} = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) = 1, \quad l(D + D_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X-S)^{(\pi)}$. We have the following

LEMMA 5.1. *There exists a unique morphism of varieties*

$$\alpha_{D_0}\colon V_{D_0} \to (X-S)^{(\pi)}$$

such that $\alpha_{D_0}(D)$ *is the unique effective divisor* $\mathfrak{m}$*-equivalent to* $D+D_0$ *for any* $D \in V_{D_0}$. *Moreover* α_{D_0} *is birational.*

Proof. Consider the Cartesian squares

$$\begin{array}{ccccc}
X_{\mathfrak{m}} \times V_{D_0} & \subset & X_{\mathfrak{m}} \times (X-S)^{(\pi)} & \xrightarrow{\ p\ } & X_{\mathfrak{m}} \\
{\scriptstyle q}\downarrow & & \downarrow & & \downarrow \\
V_{D_0} & \subset & (X-S)^{(\pi)} & \longrightarrow & \mathrm{spec}(k)\,.
\end{array}$$

Let $\mathcal{L}$ be the restriction to $X_{\mathfrak{m}} \times V_{D_0}$ of the invertible sheaf on $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$ that corresponds to the divisor $\mathcal{D} + p^*(D_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor. By Theorem 1.1 (c) the sheaf $q_*\mathcal{L}$ is invertible. The canonical map $q^*q_*\mathcal{L} \to \mathcal{L}$ induces a homorphism $s\colon \mathcal{O}_{X_{\mathfrak{m}} \times V_{D_0}} \to \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$. Using Remark 2.1, one can show that the pair $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ induces a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times V_{D_0})/V_{D_0}$. Applying Proposition 3.1 to this divisor, one gets the existence of α_{D_0}. For any $D \in V_{D_0}$, we have $l_{\mathfrak{m}}(D+D_0) = 1$ and $l(D+D_0-\mathfrak{m}) = 0$. So there is one and only one effective divisor $\mathfrak{m}$-equivalent to $D+D_0$, and this effective divisor is simply $\alpha_{D_0}(D)$.

We claim that α_{-D_0} is the birational inverse of α_{D_0}. We have

$$\begin{aligned}
\alpha_{D_0}^{-1}(V_{-D_0}) &= \{D \mid D \in V_{D_0},\ \alpha_{D_0}(D) \in V_{-D_0}\} \\
&= \{D \mid D \in V_{D_0},\ l_{\mathfrak{m}}(\alpha_{D_0}(D) - D_0) = 1,\ l(\alpha_{D_0}(D) - D_0 - \mathfrak{m}) = 0\} \\
&= V_{D_0} \cap \{D \mid l_{\mathfrak{m}}(D) = 1,\ l(D-\mathfrak{m}) = 0\} \\
&= V_{D_0} \cap V_0\,.
\end{aligned}$$

By Lemma 3.3 both V_{D_0} and V_0 are open and non-empty. Since $(X-S)^{(\pi)}$ is irreducible, the set $V_{D_0} \cap V_0$ is also open and non-empty, that is, $\alpha_{D_0}^{-1}(V_{-D_0})$ is open and non-empty. One can easily show that on this open set $\alpha_{-D_0} \circ \alpha_{D_0}$ is defined and is the identity. Similarly one can show $\alpha_{-D_0}^{-1}(V_{D_0})$ is open and non-empty, and on it $\alpha_{D_0} \circ \alpha_{-D_0}$ is defined and is the identity. So α_{D_0} is birational.

We have a birational map $\varphi\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ by the construction of $J_{\mathfrak{m}}$. Let $\mathrm{dom}(\varphi)$ be an open subset of $(X-S)^{(\pi)}$ such that $\varphi|_{\mathrm{dom}(\varphi)}$ is an open immersion, Moreover we may assume that for any $a \in \mathrm{dom}(\varphi)$, both (a, x)

and (x,a) lie in the set U defined in Lemma 3.4 (a) if x is generic, i.e., lies in some open set. In particular, $m(a,x)$ and $m(x,a)$ are defined for generic x.

Let

$$U_{D_0} = V_{D_0} \cap \mathrm{dom}(\varphi) \cap \alpha_{D_0}^{-1}(\mathrm{dom}(\varphi)).$$

Note that U_{D_0} is open and non-empty since $(X-S)^{(\pi)}$ is irreducible and α_{D_0} is birational. Moreover $\varphi(D)$ and $\varphi(\alpha_{D_0}(D))$ are defined for any $D \in U_{D_0}$. Define

$$\theta_0(D_0) = \varphi(\alpha_{D_0}(D)) - \varphi(D).$$

LEMMA 5.2. *$\theta_0(D_0)$ does not depend on the choice of D.*

Proof. Let D_1 and D_2 be two elements in U_{D_0}. We need to show that

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

Choose $D_3 \in U_{D_0}$ so that $(\alpha_{D_0}(D_1), D_3)$, $(D_1, \alpha_{D_0}(D_3))$, $(\alpha_{D_0}(D_2), D_3)$ and $(D_2, \alpha_{D_0}(D_3))$ all lie in the set U defined in Lemma 3.4 (a). Such a D_3 exists. Indeed, if $(\alpha_{D_0}(D_1), x)$, (D_1, x), $(\alpha_{D_0}(D_2), x)$ and (D_2, x) all lie in U for x lying in an open set O, then we may choose D_3 to be any element in $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$. Note that $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$ is not empty since α_{D_0} is birational and $(X-S)^{(\pi)}$ is irreducible.

We have

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(m(\alpha_{D_0}(D_1), D_3)),$$
$$\varphi(D_1) + \varphi(\alpha_{D_0}(D_3)) = \varphi(m(D_1, \alpha_{D_0}(D_3))).$$

Since

$$m(\alpha_{D_0}(D_1), D_3) \sim_{\mathrm{m}} \alpha_{D_0}(D_1) + D_3 - \pi P_0 \sim_{\mathrm{m}} D_1 + D_0 + D_3 - \pi P_0,$$
$$m(D_1, \alpha_{D_0}(D_3)) \sim_{\mathrm{m}} D_1 + \alpha_{D_0}(D_3) - \pi P_0 \sim_{\mathrm{m}} D_1 + D_3 + D_0 - \pi P_0,$$

we have

$$m(\alpha_{D_0}(D_1), D_3) = m(D_1, \alpha_{D_0}(D_3)).$$

Hence

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(D_1) + \varphi(\alpha_{D_0}(D_3)),$$

that is,

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Similarly we have

$$\varphi(\alpha_{D_0}(D_2)) - \varphi(D_2) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Therefore

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

This proves the lemma.

Thus we have a well-defined map $\theta_0\colon \mathrm{Div}^{(0)} \to J_{\mathfrak{m}}$ from the set of divisors of degree 0 on X prime to S to $J_{\mathfrak{m}}$.

LEMMA 5.3. *θ_0 is a homomorphism.*

Proof. Let $D_0, E_0 \in \mathrm{Div}^{(0)}$ and let $F_0 = D_0 + E_0$. Choose $D \in U_{D_0}$, $E \in U_{E_0}$ and $F \in U_{F_0}$ so that

$$\big(\alpha_{D_0}(D), \alpha_{E_0}(E)\big),\ (D, E),\ \big(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F\big) \text{ and } \big(m(D, E), \alpha_{F_0}(F)\big)$$

all lie in the set U defined in Lemma 3.4 (a). We have

$$\begin{aligned}\alpha_{D_0}(D) + \alpha_{E_0}(E) + F &\sim_{\mathfrak{m}} D + D_0 + E + E_0 + F = D + E + F + D_0 + E_0\,,\\ D + E + \alpha_{F_0}(F) &\sim_{\mathfrak{m}} D + E + F + F_0 = D + E + F + D_0 + E_0\,.\end{aligned}$$

So

$$m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F) = m(m(D, E), \alpha_{F_0}(F))\,.$$

Hence

$$\varphi(m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F)) = \varphi(m(m(D, E), \alpha_{F_0}(F))\,.$$

Therefore

$$\varphi(\alpha_{D_0}(D)) + \varphi(\alpha_{E_0}(E)) + \varphi(F) = \varphi(D) + \varphi(E) + \varphi(\alpha_{F_0}(F))\,,$$

or equivalently,

$$(\varphi(\alpha_{D_0}(D)) - \varphi(D)) + (\varphi(\alpha_{E_0}(E)) - \varphi(E)) = \varphi(\alpha_{F_0}(F)) - \varphi(F)\,.$$

This last equality is exactly

$$\theta_0(D_0) + \theta_0(E_0) = \theta_0(D_0 + E_0)\,.$$

So θ_0 is a homomorphism.

We define $\theta\colon \mathrm{Div} \to J_{\mathfrak{m}}$ from the group of divisors on X prime to S to $J_{\mathfrak{m}}$ by

$$\theta(D) = \theta_0(D - \deg(D)P_0)\,.$$

Obviously θ is a homomorphism.

PROPOSITION 5.4. *The homomorphism θ is surjective and* $\ker(\theta)$ *consists of divisors* $\mathfrak{m}$*-equivalent to integral multiples of* P_0.

Proof. Assume $\sum_{i=1}^{\pi} P_i$ is in $\mathrm{dom}(\varphi)$. We have

$$\theta\Big(\sum_{i=1}^{\pi} P_i\Big) = \theta_0\Big(\sum_{i=1}^{\pi} P_i - \pi P_0\Big) = \varphi(\alpha_{D_0}(D)) - \varphi(D),$$

where $D_0 = \sum_{i=1}^{\pi} P_i - \pi P_0$ and $D \in U_{D_0}$. We may choose D so that $m(\sum_{i=1}^{\pi} P_i, D)$ is defined and is the unique effective divisor $\mathfrak{m}$-equivalent to $\sum_{i=1}^{\pi} P_i + D - \pi P_0$. Since $\alpha_{D_0}(D)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D + D_0 = D + \sum_{i=1}^{\pi} P_i - \pi P_0$, we have $m(\sum_{i=1}^{\pi} P_i, D) = \alpha_{D_0}(D)$. Hence $\varphi(m(\sum_{i=1}^{\pi} P_i, D)) = \varphi(\alpha_{D_0}(D))$. So $\varphi(\sum_{i=1}^{\pi} P_i) + \varphi(D) = \varphi(\alpha_{D_0}(D))$. Therefore $\varphi(\alpha_{D_0}(D)) - \varphi(D) = \varphi(\sum_{i=1}^{\pi} P_i)$, that is,

$$\theta\Big(\sum_{i=1}^{\pi} P_i\Big) = \varphi\Big(\sum_{i=1}^{\pi} P_i\Big).$$

This is true whenever $\sum_{i=1}^{\pi} P_i$ is in $\mathrm{dom}(\varphi)$.

Since $\varphi|_{\mathrm{dom}(\varphi)}$ is an open immersion, $\varphi(\mathrm{dom}(\varphi))$ is an open subset of $J_{\mathfrak{m}}$. The image of θ contains this open subset. But $J_{\mathfrak{m}}$ is generated by any open subset. So we must have $\mathrm{Im}(\theta) = J_{\mathfrak{m}}$ and θ is surjective.

Assume $E \in \ker(\theta)$. Then $\theta_0(E - \deg(E)P_0) = 0$. Put $E_0 = E - \deg(E)P_0$. Then for any $F \in U_{E_0}$, we have

$$\varphi(\alpha_{E_0}(F)) - \varphi(F) = \theta_0(E - \deg(E)P_0) = 0.$$

Hence $\varphi(\alpha_{E_0}(F)) = \varphi(F)$. But φ is an open immersion on $\mathrm{dom}(\varphi)$. So we have $\alpha_{E_0}(F) = F$. Since $\alpha_{E_0}(F) \sim_{\mathfrak{m}} F + E_0$, we have $F \sim_{\mathfrak{m}} F + E_0$. Hence $E_0 \sim_{\mathfrak{m}} 0$, that is, $E \sim_{\mathfrak{m}} \deg(E)P_0$. So E is $\mathfrak{m}$-equivalent to an integral multiple of P_0.

Conversely assume E is $\mathfrak{m}$-equivalent to an integral multiple of P_0 and let us prove that $\theta(E) = 0$. Again let $E_0 = E - \deg(E)P_0$. Then $E_0 \sim_{\mathfrak{m}} 0$. Choose $F \in U_{E_0} \cap U_0$, where U_0 is the set U_{D_0} defined before by taking $D_0 = 0$. We have

$$\theta(E) = \theta_0(E_0) = \varphi(\alpha_{E_0}(F)) - \varphi(F),$$
$$\theta(0) = \varphi(\alpha_0(F)) - \varphi(F).$$

Note that $F + E_0 \sim_{\mathfrak{m}} F$ since $E_0 \sim_{\mathfrak{m}} 0$. But $\alpha_{E_0}(F)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $F + E_0$, and $\alpha_0(F)$ is the unique effective divisor $\mathfrak{m}$-equivalent to F. So we must have $\alpha_{E_0}(F) = \alpha_0(F)$. Therefore $\theta(E) = \theta(0) = 0$.

Regarding a point P in $X-S$ as a divisor, we can calculate $\theta(P)$. In this way we get a map $\theta\colon X-S \to J_{\mathfrak{m}}$.

PROPOSITION 5.5. *The map $\theta\colon X-S \to J_{\mathfrak{m}}$ is a morphism of algebraic varieties.*

Proof. Let $P \in X-S$ and let $D_0 = P - P_0$. Fix a $D \in U_{D_0}$. Consider the set $W_1 = \{R \in X-S \mid l_{\mathfrak{m}}(D+R-P_0) = 1\}$. By the Riemann-Roch theorem, for any R in $X-S$, we have $l_{\mathfrak{m}}(D+R-P_0) \geq 1$. Applying Theorem 1.1 (b) to the projection $q\colon X_{\mathfrak{m}} \times (X-S) \to X-S$ and the invertible sheaf corresponding to the divisor $\mathcal{D} + p^*(D-P_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor on $X_{\mathfrak{m}} \times (X-S)$ and $p\colon X_{\mathfrak{m}} \times (X-S) \to X_{\mathfrak{m}}$ is another projection, we see that W_1 is open in $X-S$. Similarly one can show $W_2 = \{R \in X-S \mid l(D+R-P_0-\mathfrak{m}) = 0\}$ is also open in $X-S$. So

$$W = W_1 \cap W_2 = \{R \in X-S \mid l_{\mathfrak{m}}(D+R-P_0) = 1, \quad l(D+R-P_0-\mathfrak{m}) = 0\}$$

is open in $X-S$. It is non-empty since $P \in W$ by our choice of D. By Proposition 3.1 we have a morphism $\gamma\colon W \to (X-S)^{(\pi)}$ of algebraic varieties such that for every $R \in W$, $\gamma(R)$ is the unique effective divisor that is $\mathfrak{m}$-equivalent to $D+R-P_0$. Since $\alpha_{R-P_0}(D)$ is the unique effective divisor that is $\mathfrak{m}$-equivalent to $D+R-P_0$, we have $\gamma(R) = \alpha_{R-P_0}(D)$. Replacing W by an open subset containing P, we may assume $\mathrm{Im}(\gamma) \subset \mathrm{dom}(\varphi)$. Note that for any $R \in W$, we have $D \in U_{R-P_0}$, and

$$\theta(R) = \theta_0(R-P_0) = \varphi((\alpha_{R-P_0}(D)) - \varphi(D) = \varphi(\gamma(R)) - \varphi(D),$$

that is, $\theta(R) = \varphi(\gamma(R)) - \varphi(D)$. So $\theta = \varphi \circ \gamma - \varphi(D)$ on W. This proves θ is a morphism of algebraic varieties in an open subset containing P. Since $P \in X-S$ is arbitrary, θ is a morphism of algebraic varieties.

The morphism $\theta\colon X-S \to J_{\mathfrak{m}}$ induces a morphism of algebraic varieties $\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$.

PROPOSITION 5.6. *$\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ coincides with the birational map $\varphi\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$. In particular φ is everywhere defined.*

Proof. Let $\sum_{i=1}^{\pi} P_i \in \mathrm{dom}(\varphi)$. By the proof of Proposition 5.4, we have $\varphi(\sum_{i=1}^{\pi} P_i) = \theta(\sum_{i=1}^{\pi} P_i)$. So $\varphi = \theta$ as rational maps.

Thus there is no difference between φ and θ. From now on we denote the map φ also by θ. We summarize what we have so far in the following theorem.

THEOREM 1. *There is a morphism of algebraic varieties* $\theta\colon X - S \to J_{\mathfrak{m}}$ *satisfying the following properties:*

(a) *The extension of* θ *to the group of divisors on* X *prime to* S *induces, by passing to quotient, an isomorphism between the group* $C^0_{\mathfrak{m}}$ *of classes of divisors of degree zero with respect to* $\mathfrak{m}$*-equivalence and the group* $J_{\mathfrak{m}}$.

(b) *The extension of* θ *to* $(X - S)^{(\pi)}$ *induces a birational map from* $X^{(\pi)}$ *to* $J_{\mathfrak{m}}$.

The following theorem characterizes $J_{\mathfrak{m}}$ by a universal property:

THEOREM 2. *Let* $f\colon X \to G$ *be a rational map from* X *to a commutative algebraic group* G *and assume* $\mathfrak{m}$ *is a modulus for* f. *Then there is a unique homomorphism* $F\colon J_{\mathfrak{m}} \to G$ *of algebraic groups such that* $f = F \circ \theta + f(P_0)$.

Proof. Replacing f by $f - f(P_0)$, we may assume $f(P_0) = 0$. Since $\mathfrak{m}$ is a modulus for f, the extension of f to the group of divisors of X prime to S induces a homomorphism $C^0_{\mathfrak{m}} \to G$ by passing to quotient. By Theorem 1 (a) we have $J_{\mathfrak{m}} \cong C^0_{\mathfrak{m}}$ as groups. So we have a homomorphism of groups $F\colon J_{\mathfrak{m}} \to G$ such that $f = F\theta$. It remains to prove F is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map $\theta\colon (X - S)^{(\pi)} \to J_{\mathfrak{m}}$. Denote the extension of f to $(X - S)^{(\pi)}$ by f'. Then $F\theta = f'$. Since θ is birational, it induces an isomorphism between an open subvariety of $(X - S)^{(\pi)}$ and an open subvariety of $J_{\mathfrak{m}}$. Moreover f' is a morphism of algebraic varieties. Hence F is a morphism of algebraic varieties when restricted to some open subset of $J_{\mathfrak{m}}$. The whole $J_{\mathfrak{m}}$ can be obtained from this open subset by translation. So F is a morphism of algebraic varieties.

6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove $J_{\mathfrak{m}}$ is the Picard scheme of $X_{\mathfrak{m}}$.

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k)\,. \end{array}$$

We have $q_*\mathcal{O}_{X_{\mathfrak{m}} \times T} = \mathcal{O}_T$ by [EGA] III, § 1.4.15, the fact $H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}) = k$, and the fact that $T \to \operatorname{spec}(k)$ is flat. The morphism q has a section $s\colon T \to X_{\mathfrak{m}} \times T$, $t \mapsto (P_0, t)$.