Appendix : Linearization of the Lagrange top on an elliptic curve

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APPENDIX: LINEARIZATION OF THE LAGRANGE TOP ON AN ELLIPTIC CURVE

The purpose of the present Appendix is to give a brief account of some "well known" facts concerning the linearization of the Lagrange top on an elliptic curve. All algebraic varieties below come equipped with real structures. We shall make the following convention. If the complex algebraic varieties V_1 and V_2 are isomorphic over $\mathbf{R}$, then we shall simply write $V_1 = V_2$.

Further we shall suppose that the invariant complex level set

$$T_h = \left\{ (\Omega, \Gamma) \in \mathbf{C}^6 \ : \ H_1 = 1\,,\ H_2 = h_2\,,\ H_3 = h_3\,,\ H_4 = h_4 \right\}$$

of the Lagrange top (2) is smooth, and moreover $h = (h_2, h_3, h_4) \in \mathbf{R}^3$. Thus T_h has a natural real structure, and if $T_h^{\mathbf{R}}$ is its real part we make the assumption $T_h^{\mathbf{R}} \neq \varnothing$. Recall that to T_h we associate the following smooth algebraic curves:

(i) the Lagrange curve $\Gamma_h = \{\eta^2 = 4\xi^3 - g_2\xi - g_3\}$ where $g_2 = g_2(h)$, $g_3 = g_3(h)$ are given by (53) and (23). The polynomial $4\xi^3 - g_2\xi - g_3$ has three real roots, so the curve Γ_h has two ovals. Denote by $\overline{\Gamma}_h$ the completed curve Γ_h.

(ii) the spectral curve $\widetilde{C}_h = \{\mu^2 + f(\lambda) = 0\}$ of the Lax pair of Adler and van Moerbeke (26), with the natural anti-holomorphic involution $(\lambda, \mu) \mapsto (\overline{\lambda}, \overline{\mu})$, where $f(\lambda)$ is given by (24). It is isomorphic over $\mathbf{R}$ to the curve $C_h = \{\mu^2 = f(\lambda)\}$ with an anti-holomorphic involution $(\lambda, \mu) \mapsto (\overline{\lambda}, -\overline{\mu})$, so $\widetilde{C}_h = C_h$. The polynomial $f(\lambda)$ has two pairs of complex conjugate roots.

(iii) the Jacobian $J(C_h) = \mathrm{Pic}^2(C_h)$ of C_h which is identified, via the Euler-Weil map ([25]), to the Lagrange curve $\overline{\Gamma}_h$, so $J(C_h) = \overline{\Gamma}_h$.

According to the context the curves $\widetilde{C}_h$, C_h will be considered either as affine, or as completed and normalized curves.

Recall also that the generalized Jacobian $J(C_h; \infty^{\pm}) = \mathbf{C}^2/\Lambda$ of the elliptic curve C_h with two points identified is defined as an extension of $J(C_h)$ by C^*

$$0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{\iota} J(C_h; \infty^{\pm}) \xrightarrow{\phi} J(C_h) \to 0\,.$$

By Theorem 2.2 the invariant complex level set T_h identifies with $J(C_h; \infty^{\pm}) - D_\infty$, where $D_\infty = \phi^{-1}(p)$, $p = \infty \in \Gamma_h$, so we obtain the following exact sequence

$$0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{\iota} T_h \xrightarrow{\phi} \Gamma_h \to 0\,. \tag{61}$$

Denote by $\bar{T}_h$ the variety T_h completed by the curve D_∞, so $\bar{T}_h = J(C_h; \infty^\pm)$. It follows from Theorem 3.6 that a point $t \in J(C_h)$ is defined by $\Gamma_3(t)$ and its derivative in t, and hence

$$\begin{aligned}
&\phi\colon T_h \to \Gamma_h : (\Omega, \Gamma) \mapsto (\eta, \xi) \\
&\xi = -\tfrac{1}{2}\Gamma_3, \qquad \eta = -\tfrac{1}{2}\tfrac{d}{dt}\Gamma_3(t) = -\tfrac{1}{2}(\Gamma_1\Omega_2 - \Gamma_2\Omega_1)\,.
\end{aligned} \tag{62}$$

The map ι in (61) defines a $\mathbf{C}^*$-action on T_h which is just the action of the linear complex flow of (3). The latter is obviously given by

$$\begin{aligned}
&\Omega_1 \pm i\Omega_2 \mapsto e^{\pm b}(\Omega_1 \pm i\Omega_2)\,, \qquad (M_3, \Gamma_3) \mapsto (M_3, \Gamma_3) \\
&\Gamma_1 \pm i\Gamma_2 \mapsto e^{\pm b}(\Gamma_1 \pm i\Gamma_2)\,, \qquad e^b \in \mathbf{C}^*\,.
\end{aligned} \tag{63}$$

This C^*-action is free and compatible with the projection map ϕ so we have a well defined quotient map

$$\phi\colon T_h/\mathbf{C}^* \to \Gamma_h\,,$$

which is an isomorphism. It is obviously prolonged to the isomorphism

$$\phi\colon \bar{T}_h/\mathbf{C}^* \to \bar{\Gamma}_h\,.$$

As Γ_3 is a first integral of (3), then the corresponding flow is projected on $\bar{\Gamma}_h$ to the identity. According to Theorem 3.6 we have $\Gamma_3(t) = -2\wp(t) + \text{constant}$, and hence the flow of the Lagrange top is projected to a linear flow on the Lagrange curve Γ_h. The real part of T_h is a torus $T_h^{\mathbf{R}} \sim S_1 \times S_1$ on which the real flow of (3) defines a free circle action $\Re = S^1$ compatible with ϕ. $T_h^{\mathbf{R}}$ is compact and connected so is $\phi(T_h^{\mathbf{R}})$. It follows that $\phi\big(T_h^{\mathbf{R}}\big) = \phi\big(T_h^{\mathbf{R}}/\Re\big)$ is contained in the compact oval of the Lagrange curve Γ_h. In fact, ϕ provides an isomorphism between $T_h^{\mathbf{R}}/\Re$ and this oval. Indeed, the only thing we need to check is that the pre-image of a point on this compact oval, under the map $\phi\colon T_h^{\mathbf{R}} \to \Gamma_h$, is a single orbit of the system (3), that is to say a circle. But a point t on Γ_h is determined by $\Gamma_3(t)$ and $\frac{d}{dt}\Gamma_3(t) = \Gamma_1\Omega_2 - \Gamma_2\Omega_1$. This combined with the first integrals amounts to fixing Ω_3, Γ_3, the lengths

$$\Omega_1^2 + \Omega_2^2\,, \qquad \Gamma_1^2 + \Gamma_2^2\,,$$

the scalar product

$$\Omega_1\Gamma_1 + \Omega_2\Gamma_2$$

and the vector product

$$\Gamma_1\Omega_2 - \Gamma_2\Omega_1$$

of the real vectors (Ω_1, Ω_2), (Γ_1, Γ_2), which defines a circle. To sum up, we have

THEOREM A.1 (Lagrange linearization).
(i) $\phi\colon T_h/\mathbf{C}^* \to \Gamma_h$ *is an isomorphism.*

(ii) $\phi\colon \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h$ *is an isomorphism.*

(iii) *The image of the flow of* (3) *on* $\overline{\Gamma}_h$ *is the identity, and that of* (2) *is linear.*

(iv) *The map* ϕ *provides an isomorphism between* $T_h^{\mathbf{R}}/\mathfrak{R}$ *and the compact oval of the affine real curve* Γ_h.

The above theorem may be attributed to Lagrange [17, p. 254] who computed the differential equation satisfied by the nutation $\Gamma_3(t)$. It worth noting that this computation was published in 1813 (the year when Lagrange died) by Poisson [19] as completely new, and without mentioning Lagrange.

There is another more sophisticated way to linearize the Lagrange top on the elliptic curve Γ_h, by making use of the Lax pair representation (26) (see [1, 21, 24, 2, 3])

$$\frac{d}{dt}\left(\lambda^2\chi + \lambda M - \Gamma\right) = \left[\lambda^2\chi + \lambda M - \Gamma\,,\ \lambda\chi + \Omega\right].$$

Namely, let $\overset{\circ}{C}_h$ be the affine curve $\widetilde{C}_h$ with its Weierstrass points removed (they correspond to the roots of $f(\lambda)$), and put $A(\lambda) = \lambda^2\chi + \lambda M - \Gamma$. As $-\mu\left(\mu^2 + f(\lambda)\right) = \det(A(\lambda) - \mu I)$, then for $(\lambda, \mu) \in \overset{\circ}{C}_h$ we have $\dim\operatorname{Ker}\left(A(\lambda) - \mu I\right) = 1$. It follows that the variety

$$\left\{(\lambda,\mu) \in \overset{\circ}{C},\ [v_0, v_1, v_2] \in \mathbf{CP}^2 : (v_0, v_1, v_2) \in \operatorname{Ker}(A(\lambda) - \mu I)\right\} \subset \widetilde{C}_h \times \mathbf{CP}^2$$

is smooth and it is easy to check that its closure in $\left\{\widetilde{C}_h \cup \infty^+ \cup \infty^-\right\} \times \mathbf{CP}^2$ is also smooth, so we have a holomorphic line bundle on the compactified and normalized curve $\left\{\widetilde{C}_h \cup \infty^+ \cup \infty^-\right\}$ (this also follows from [12, Proposition 2.2]). One computes further that the degree of this bundle is 4 and there is always a meromorphic section with a pole divisor $D = R_+ + R_- + \infty^+ + \infty^-$. Of course, the divisor D depends on the coefficients of the polynomial matrix $A(\lambda)$, and hence on (Ω, Γ). Consider now the map

$$\begin{aligned}\widetilde{\phi}\colon T_h &\to \operatorname{Pic}^2(\widetilde{C}_h) = J(\widetilde{C}_h) = \overline{\Gamma}_h\\ (\Omega,\Gamma) &\mapsto [R_+ + R_-]\end{aligned}$$

where the divisor $R_\pm = \left(\lambda(R_\pm), \mu(R_\pm)\right) \in \widetilde{C}_h$ is equal to

$$\lambda(R_\pm) = \frac{\Gamma_1 \mp i\Gamma_2}{\Omega_1 \mp i\Omega_2}, \qquad \mu(R_\pm) = \pm i\left(-\Gamma_3 + (1+m)h_4\lambda(R_\pm) + \lambda^2(R_\pm)\right).$$

Note that, according to Theorem 3.6, the map $\widetilde{\phi}$ is prolonged to a holomorphic map

$$\widetilde{\phi}: \overline{T}_h \to \mathrm{Pic}^2(\widetilde{C}_h) = J(\widetilde{C}_h) = \overline{\Gamma}_h .$$

We shall show that the map $\widetilde{\phi}$ provides a linearization of the Lagrange top on $\overline{\Gamma}_h$. It is obvious that $\widetilde{\phi}$ is compatible with the $\mathbf{C}^*$ action (63) on $T_h, \widetilde{T}_h$, so we have the holomorphic maps

$$\widetilde{\phi}: \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h , \qquad \phi: \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h , \qquad \widetilde{\phi} \circ \phi^{-1}: \overline{\Gamma}_h \to \overline{\Gamma}_h .$$

Remembering that $\overline{\Gamma}_h$ is a complex torus, we conclude that if $z \in \mathbf{C}/\Lambda \sim \overline{\Gamma}_h$, then $\widetilde{\phi} \circ \phi^{-1}(z) = kz$, for some $k \in \mathbf{Z}$, and hence $\widetilde{\phi}$ provides a linearization on $\overline{\Gamma}_h$ too. The map $\widetilde{\phi}$ is a non-ramified covering of degree k^2 and it is easy to check that $k^2 = 4$. Indeed, if $R_+ + R_-$ is linearly equivalent on $\widetilde{C}_h$ to $\infty^+ + \infty^-$, then $R_+ = \sigma(R_-)$, where $\sigma(\lambda, \mu) = (\lambda, -\mu)$ is the elliptic involution. It follows that

$$\frac{\Gamma_1 + i\Gamma_2}{\Omega_1 + i\Omega_2} = \frac{\Gamma_1 - i\Gamma_2}{\Omega_1 - i\Omega_2} \iff \Omega_1\Gamma_2 - \Omega_2\Gamma_1 = \frac{d}{dt}\Gamma_3(t) = 0 ,$$

which shows that the pre-image of the divisor class $\infty^+ + \infty^-$ on $\overline{\Gamma}_h$ with respect to $\widetilde{\phi} \circ \phi^{-1}$ consists of the four Weierstrass points on $\overline{\Gamma}_h$. Finally, we note that $\widetilde{\phi}(T_h^{\mathbf{R}}/\Re)$, as before, is contained in an oval of $\overline{\Gamma}_h$. In this case, however, $\widetilde{\phi}$ provides a double non-ramified covering of $T_h^{\mathbf{R}}/\Re$ to its image – the oval of the curve $\overline{\Gamma}_h = \mathrm{Pic}^2(\widetilde{C}_h)$ containing the point ∞. Indeed, note that the divisor class of $\infty^+ + \infty^-$ represents a real point on $\mathrm{Pic}^2(\widetilde{C}_h)$. It has exactly two real pre-images: the two Weirstrass points contained in the compact oval of Γ_h, and the remaining two Weirstrass points are not real. Thus we have proved the following

THEOREM A.2 (Linearization by making use of a Lax pair). *Let Γ_h be the affine curve defined above, and*

$$\overset{\circ}{T}_h = T_h \setminus \{(\omega, \Gamma) \in \mathbf{C}^6 \; : \; \Omega_1\Gamma_2 - \Omega_2\Gamma_1 = 0\} .$$

Then

(i) $\widetilde{\phi}: \overset{\circ}{T}_h/\mathbf{C}^* \to \Gamma_h$ *is a non-ramified covering of degree* 4.

(ii) $\widetilde{\phi}: \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h$ *is a non-ramified covering of degree* 4.

(iii) *The image of the flow of* (3) *on* $\overline{\Gamma}_h$ *is the identity, and that of* (2) *is linear.*

(iv) *The map* $\widetilde{\phi}$ *provides a double non-ramified covering of* $T_h^{\mathbf{R}}/\Re$ *to its image – the oval of the compactified and normalized curve* $\overline{\Gamma}_h = \Gamma_h \cup \infty$ *containing the point* ∞.

Statement (iv) is due to M. Audin. In this form it appeared first in [2] (Proposition 3.3.2) but the proof is not correct. Earlier Verdier [24] wrongly claimed that the map $\widetilde{\phi}$ provides an isomorphism between $T_h^{\mathbf{R}}/\Re$ and its image. Statement (iii) is a well known fact, though it does not seem to have ever been rigorously proved. Thus Adler and van Moerbeke [1] and then Ratiu and van Moerbeke [21] proposed a "proof" based on a general scheme for linearizing the flow defined by a Lax pair with a spectral parameter (e.g. Adler and van Moerbeke [1], Theorem 1, p. 337). The Lax pair (26) does not fit, however, the general procedure, as its spectral curve is always reducible. Of course this is only a minor technical difficulty as we may also use the Lax pair (14). It was proposed in [1, p. 351] and [21] to consider, instead of the Lax pair (26), another Lax pair

$$\frac{dA^\epsilon}{dt} = [A^\epsilon, B^\epsilon]\,, \tag{64}$$

where in the notation of [1] we have

$$A^\epsilon = A^\epsilon(h) = \begin{pmatrix} \epsilon h^2 & \beta & i\beta^* \\ -\beta^* & -\omega & 0 \\ i\beta & 0 & \omega \end{pmatrix}, \qquad B^\epsilon = B^\epsilon(h) = \frac{1}{I_1}\,[h^{-1}A^\epsilon(h)]_+\,,$$

$[\,.\,]_+$ means "polynomial part" and

$$\begin{aligned} \beta &= y + hx\,, & y &= \tfrac{1}{\sqrt{2}}\,(\gamma_1 - i\gamma_2)\,, & x &= \tfrac{I_1}{\sqrt{2}}\,(\Omega_1 - i\Omega_2) \\ \beta^* &= \bar{y} + h\bar{x}\,, & \bar{y} &= \tfrac{1}{\sqrt{2}}\,(\gamma_1 + i\gamma_2)\,, & \bar{x} &= \tfrac{I_1}{\sqrt{2}}\,(\Omega_1 + i\Omega_2) \end{aligned}$$

$$i\omega = z_0 I_1 h^2 + I_3\Omega_3 h + \gamma_3\,.$$

To obtain our notations from those of [1], we just replace

$$\gamma_i = -\Gamma_i\,, \quad z_0 = I_1 = 1\,, \quad I_3 = 1 + m\,, \quad h = \lambda\,.$$

For the spectral curve X_ϵ of A^ϵ we obtain

$$\begin{aligned} \det\big(A^\epsilon(h) - zI\big) &= (\epsilon h^2 - z)(z^2 - \omega^2) - 2\beta\beta^* z \\ &= -z^3 + \epsilon h^2 z^2 + (-2\beta\beta^* + \omega^2)z - \epsilon h^2\omega^2 = 0\,. \end{aligned} \tag{65}$$

This is generically a smooth irreducible genus 4 curve, so the Lax pair (64) fits to Theorem 1, p. 337 in [1]. Thus the flow of (64) linearizes on $\mathrm{Jac}(X_\epsilon)$ and when $\epsilon \to 0$ it goes over into a linear flow on the compact piece of $\mathrm{Jac}(X_0)$ which is just the Lagrange elliptic curve. On the other hand the differential equation (64) for $\epsilon = 0$ is, modulo a linear change of the variables, the original system (2) which establishes once again Theorem A.2, (ii). It is easy to see, however, that the above approach does not work as for $\epsilon \neq 0$ the Lax pair (64) *does not define a differential equation.* Indeed, note that (64) is equivalent to the Lax pair

$$\frac{dA^0}{dt} = [A^0, B^0] - \frac{\epsilon h}{I_1}\begin{pmatrix} 0 & y & i\bar{y} \\ -\bar{y} & i\gamma_3 & 0 \\ iy & 0 & -i\gamma_3 \end{pmatrix}. \tag{66}$$

Its $(1,2)$ entry is computed to be

$$\frac{d\beta}{dt} = \frac{i}{I_1}\left(yI_3\Omega_3 - x\gamma_3 + hz_0I_1y\right) - \frac{\epsilon h y}{I_1}$$

and the $(3,1)$ entry is

$$i\frac{d\beta}{dt} = \frac{1}{I_1}\left(-yI_3\Omega_3 + x\gamma_3 - hz_0I_1y\right) + \frac{\epsilon^3 h y}{I_1},$$

so $y \equiv 0$ and in a similar way $\bar{y} \equiv 0$.

More generally, it is seen from the coefficients of the spectral curve X_ϵ, $\epsilon \neq 0$, that the functions

$$\Omega_1^2 + \Omega_2^2, \qquad \gamma_1^2 + \gamma_2^2, \qquad \Omega_1\gamma_1 + \Omega_2\gamma_2, \qquad \gamma_3, \qquad \Omega_3$$

are invariants for *any* isospectral deformation of the matrix A^ϵ. By continuity these five functions are invariants for $\epsilon = 0$ too, so the vector field in $\mathbf{C}^6$ obtained as $\epsilon \to 0$ is collinear to the linear vector field of (3). Of course there is no analytic change of variables in $\mathbf{C}^6$ which sends the orbits of (3) to orbits of (2).

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