

# A. Appendix

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$$\begin{aligned}
&= \left( [n-j+1] - q^{1/2}[n-j] - q^{-1/2}[n-j] + [n-j-1] \right) \left\langle \begin{array}{c} 1 \uparrow \\ \text{---} j-1 \text{---} \\ 1 \downarrow \end{array} \begin{array}{c} j \\ \text{---} j \text{---} \\ j \downarrow \end{array} \right\rangle_n \\
&\quad + \left\langle \begin{array}{c} 1 \uparrow \\ \text{---} \\ 1 \downarrow \end{array} \begin{array}{c} j \\ \text{---} \\ j \downarrow \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} 1 \uparrow \\ \text{---} \\ 1 \downarrow \end{array} \begin{array}{c} j \\ \text{---} \\ j \downarrow \end{array} \right\rangle_n,
\end{aligned}$$

where we use Lemma 5.2 in the third equality. Now the proof for the Reidemeister move II is complete.

*The invariance under the Reidemeister move III.* This is proved by repeated application of Lemma 5.3 and details are omitted. See the proof of Theorem 3.1.

#### A. APPENDIX

In this appendix, we give proofs of lemmas used in the previous section.

LEMMA A.1.

$$\left\langle \begin{array}{c} i \uparrow \\ \text{---} j \text{---} i-j \\ i \downarrow \end{array} \right\rangle_n = [j] \left\langle \begin{array}{c} i \uparrow \\ \text{---} \\ i \downarrow \end{array} \right\rangle_n$$

for  $i \geq j \geq 0$ .

*Proof.* The proof for  $j = 1$  is similar to that of Lemma 2.2 and omitted. For  $j > 1$  we have

$$\begin{aligned}
\left\langle \begin{array}{c} i \uparrow \\ \text{---} j \text{---} i-j \\ i \downarrow \end{array} \right\rangle_n &= \frac{1}{[j]} \left\langle \begin{array}{c} i \uparrow \\ \text{---} j-1 \text{---} 1 \text{---} i-j \\ j \uparrow \text{---} j \downarrow \end{array} \right\rangle_n = \frac{1}{[j]} \left\langle \begin{array}{c} i \uparrow \\ \text{---} j-1 \text{---} 1 \text{---} i-j+1 \\ j \uparrow \text{---} j \downarrow \end{array} \right\rangle_n \\
&= \frac{[i-j+1]}{[j]} \left\langle \begin{array}{c} i \uparrow \\ \text{---} j-1 \text{---} i-j+1 \\ i \downarrow \end{array} \right\rangle_n = \frac{[i-j+1]}{[j]} [j-1] \left\langle \begin{array}{c} i \uparrow \\ \text{---} \\ i \downarrow \end{array} \right\rangle_n = [j] \left\langle \begin{array}{c} i \uparrow \\ \text{---} \\ i \downarrow \end{array} \right\rangle_n,
\end{aligned}$$

where the second equality follows from Lemma 2.6 and the fourth by induction. The proof is complete.  $\square$

LEMMA A.2.

$$\left\langle \begin{array}{c} i \\ j+k \\ j \end{array} \begin{array}{c} j+k-i \\ k \\ i \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \\ i-k \\ j \end{array} \begin{array}{c} k \\ j+k-i \\ i \end{array} \right\rangle_n$$

for  $j+k \geq i \geq k \geq 0$ .

*Proof.* By a  $\pi$ -rotation and orientation reversing, we get the right hand side from the left hand side. So there is a one to one correspondence between the states of both hand sides and the equality follows from the definition.  $\square$

LEMMA A.3.

$$\left\langle \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \begin{array}{c} 1 \\ j-1 \\ j \end{array} \right\rangle_n = \left\langle \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} j \\ j+1 \\ j \end{array} \right\rangle_n + [j-1] \left\langle \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} j \\ j \end{array} \right\rangle_n$$

for  $j \geq 1$ .

*Proof.* We can prove the equality directly from the definition as in the proofs in the lemmas in §2. Details are omitted.  $\square$

LEMMA A.4.

$$\left\langle \begin{array}{c} 1 \\ k+1 \\ 1 \end{array} \begin{array}{c} k \\ j-k \\ j \end{array} \right\rangle_n = \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} \left\langle \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} j \\ j+1 \\ j \end{array} \right\rangle_n + \begin{bmatrix} j-1 \\ k \end{bmatrix} \left\langle \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} j \\ j \end{array} \right\rangle_n$$

for  $j \geq k \geq 1$ .

*Proof.* From Lemma A.3 (substituting  $j$  with  $k$ ) the left hand side becomes

$$\begin{aligned}
 & \left\langle \begin{array}{c} 1 \uparrow \\ 2 \leftarrow 1 \rightarrow k \\ 1 \uparrow \end{array} \begin{array}{c} j \\ k \\ k-1 \\ j-k \\ k \\ j \end{array} \right\rangle_n - [k-1] \left\langle \begin{array}{c} 1 \uparrow \\ k \leftarrow j \rightarrow j-k \\ j \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} 1 \uparrow \\ 2 \leftarrow 1 \rightarrow j \\ 1 \uparrow \end{array} \begin{array}{c} j \\ j-1 \\ k-1 \\ j-k \\ j-1 \\ j \end{array} \right\rangle_n - [k-1] \left[ \begin{array}{c} j \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n \\
 &= \left[ \begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ 2 \leftarrow 1 \rightarrow j \\ 1 \uparrow \end{array} \begin{array}{c} j \\ j-1 \\ j-1 \\ j \end{array} \right\rangle_n - [k-1] \left[ \begin{array}{c} j \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n \\
 &= \left[ \begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ 1 \leftarrow j+1 \rightarrow j \\ 1 \uparrow \end{array} \right\rangle_n + \left( \left[ \begin{array}{c} j-1 \\ k-1 \end{array} \right] [j-1] - [k-1] \left[ \begin{array}{c} j \\ k \end{array} \right] \right) \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n \\
 &= \left[ \begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ 1 \leftarrow j+1 \rightarrow j \\ 1 \uparrow \end{array} \right\rangle_n + \left[ \begin{array}{c} j-1 \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n,
 \end{aligned}$$

where the third equality follows from Lemma A.3. The proof is complete.  $\square$

LEMMA A.5.

$$\left\langle \begin{array}{c} 2 \uparrow \\ 3 \leftarrow 1 \rightarrow j \\ 2 \uparrow \end{array} \begin{array}{c} j \\ j-1 \\ j \\ j \end{array} \right\rangle_n - \left\langle \begin{array}{c} 2 \uparrow \\ 1 \leftarrow 1 \rightarrow j+1 \\ 2 \uparrow \end{array} \begin{array}{c} j \\ j+1 \\ j \\ j \end{array} \right\rangle_n = [j-2] \left\langle \begin{array}{c} 2 \uparrow \\ \uparrow j \end{array} \right\rangle_n$$

for  $j \geq 2$ .

*Proof.* We have

$$\begin{aligned}
 \left\langle \begin{array}{c} 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n &= \frac{1}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ j-2 \uparrow \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\
 &= \frac{1}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\
 &= \frac{1}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + \frac{[j-1]}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + [j-2] \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \end{array} \right\rangle_n,
 \end{aligned}$$

where we use Lemma A.2 in the third equality and Lemma A.4 in the fourth equality. The proof is complete.  $\square$

LEMMA A.6.

$$\left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n - \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n = [j-i] \left\langle \begin{array}{c} i \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{c} j \uparrow \\ \uparrow \\ \uparrow \end{array} \right\rangle_n.$$

*Proof.* We have

$$\begin{aligned} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n &= \frac{1}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ \text{diamond} \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\ &= \frac{1}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} \begin{array}{c} \uparrow j \\ \uparrow 2 \\ \uparrow 1 \\ \uparrow 2 \\ \uparrow j \end{array} \\ j-2 \uparrow \end{array} \right\rangle_n \\ &= \frac{1}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} \begin{array}{c} \uparrow j \\ \uparrow 2 \\ \uparrow 3 \\ \uparrow 2 \\ \uparrow j \end{array} \\ j-2 \uparrow \end{array} \right\rangle_n - \frac{[i-2]}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{c} \begin{array}{c} \uparrow j \\ \uparrow 2 \\ \text{diamond} \\ \uparrow 2 \\ \uparrow j \end{array} \\ j-2 \uparrow \end{array} \right\rangle_n \\ &= \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + \left( \frac{[j-2][i]}{[2]} - \frac{[i-2][j]}{[j-1]} \right) \left\langle \begin{array}{c} i \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{c} j \uparrow \\ \uparrow \\ \uparrow \end{array} \right\rangle_n \end{aligned}$$

$$= \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \\ j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + [j-i] \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \end{array} \right\rangle_n,$$

where we use Lemmas A.3 and A.4 in the third and the fourth equalities respectively. The proof is complete.  $\square$

LEMMA A.7.

$$\left\langle \begin{array}{c} i \uparrow \\ i+k \uparrow \\ 1 \uparrow \\ j \uparrow \\ j-k \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n = \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} \left\langle \begin{array}{c} i \uparrow \\ 1 \uparrow \\ j \uparrow \\ i+j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n + \begin{bmatrix} j-1 \\ k \end{bmatrix} \left\langle \begin{array}{c} i \uparrow \\ 1 \uparrow \\ j \uparrow \\ i-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n$$

for  $j > k \geq 1$ .

*Proof.* We first prove the case  $k = 1$ . We will show

$$\left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ 1 \uparrow \\ j \uparrow \\ j-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \uparrow \\ 1 \uparrow \\ j \uparrow \\ i+j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n + [j-1] \left\langle \begin{array}{c} i \uparrow \\ 1 \uparrow \\ j \uparrow \\ i-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n$$

by induction on  $j$ . Note that Lemma A.3 is the case  $i = 1$ .

The left hand side becomes

$$\frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \\ 1 \uparrow \\ j \uparrow \\ j-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \\ 1 \uparrow \\ j \uparrow \\ j+1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n + \frac{[j-i]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ 1 \uparrow \\ j \uparrow \\ i-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n$$

$$\begin{aligned}
&= \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \downarrow i-1 \\ \uparrow j+1 \\ \downarrow 1 \\ \uparrow i+j \\ \downarrow i+j-1 \end{array} \right\rangle_n + \frac{[j]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \downarrow i-1 \\ \uparrow j+1 \\ \downarrow 1 \\ \uparrow i-2 \\ \downarrow i+j-1 \end{array} \right\rangle_n + \frac{[j-i]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \downarrow 1 \\ \uparrow i-1 \\ \downarrow i+j-1 \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} i \uparrow \\ \downarrow 1 \\ \uparrow j \\ \downarrow i+j \\ \downarrow i+j-1 \end{array} \right\rangle_n + \frac{[j][i-1] + [j-i]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \downarrow 1 \\ \uparrow i-1 \\ \downarrow i+j-1 \end{array} \right\rangle_n,
\end{aligned}$$

which is equal to the right hand side. Here we use Lemma A.6 in the first equality and the inductive hypothesis in the second equality.

The proof for  $k > 1$  is similar and omitted.  $\square$

LEMMA A.8.

$$\left\langle \begin{array}{c} i \uparrow \\ \downarrow j+k \\ \uparrow j+1 \\ \downarrow j \\ \uparrow i-k+1 \\ \downarrow i+1 \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \uparrow \\ \downarrow i-k \\ \uparrow j+1 \\ \downarrow j \\ \uparrow j+k-i \\ \downarrow i+1 \end{array} \right\rangle_n + \left\langle \begin{array}{c} i \uparrow \\ \downarrow i-k+1 \\ \uparrow j+1 \\ \downarrow j \\ \uparrow j+k-i-1 \\ \downarrow i+1 \end{array} \right\rangle_n$$

for  $j \geq i \geq k \geq 1$ .

*Proof.* For  $k = 1$  we have

$$\begin{aligned}
\left\langle \begin{array}{c} i \uparrow \\ \downarrow j+1 \\ \uparrow j+1 \\ \downarrow j \\ \uparrow i \\ \downarrow i+1 \end{array} \right\rangle_n &= \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \downarrow j+1 \\ \uparrow j \\ \downarrow j+1 \\ \uparrow i-1 \\ \downarrow i \\ \uparrow i+1 \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \downarrow i-1 \\ \uparrow j \\ \downarrow j+1 \\ \uparrow j-i+1 \\ \downarrow i \\ \uparrow i+1 \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} i \uparrow \\ \downarrow i-1 \\ \uparrow j+1 \\ \downarrow j \\ \uparrow j-i+1 \\ \downarrow i+1 \end{array} \right\rangle_n + \left\langle \begin{array}{c} i \uparrow \\ \downarrow j \\ \uparrow j-i \\ \downarrow i+1 \end{array} \right\rangle_n,
\end{aligned}$$

where we use Lemma A.7 in the last equality.

The proof for  $k > 1$  is similar and left to the reader.  $\square$

PROPOSITION A.9.

$$\sum_{k=0}^{i+1} (-1)^k \left\langle \begin{array}{c} i \uparrow \\ j+k \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{j+k-i} \\ \xleftarrow{k} \end{array} \begin{array}{c} j+1 \uparrow \\ i-k+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n = 0.$$

*Proof.* From Lemma A.8, we see that the left hand side equals

$$\begin{aligned} & \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{j-i} \end{array} \begin{array}{c} j+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n + \sum_{k=1}^i (-1)^k \left\langle \begin{array}{c} i \uparrow \\ i-k \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xleftarrow{k} \\ \xrightarrow{j+k-i} \end{array} \begin{array}{c} j+1 \uparrow \\ j+k+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n \\ & + \sum_{k=1}^i (-1)^k \left\langle \begin{array}{c} i \uparrow \\ i-k+1 \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xleftarrow{k-1} \\ \xrightarrow{j+k-i-1} \end{array} \begin{array}{c} j+1 \uparrow \\ j+k \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n + (-1)^{i+1} \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{i+j+1} \end{array} \begin{array}{c} j+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n \\ & = \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{j-i} \end{array} \begin{array}{c} j+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n + \sum_{k=1}^i (-1)^k \left\langle \begin{array}{c} i \uparrow \\ i-k \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xleftarrow{k} \\ \xrightarrow{j+k-i} \end{array} \begin{array}{c} j+1 \uparrow \\ j+k+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n \\ & + \sum_{k=0}^{i-1} (-1)^{k+1} \left\langle \begin{array}{c} i \uparrow \\ i-k \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xleftarrow{k} \\ \xrightarrow{j+k-i} \end{array} \begin{array}{c} j+1 \uparrow \\ j+k+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n + (-1)^{i+1} \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{i+j+1} \end{array} \begin{array}{c} j+1 \uparrow \\ i+1 \uparrow \end{array} \right\rangle_n \\ & = 0, \quad \text{and the proof is complete.} \quad \square \end{aligned}$$

More generally we have the following formula, which was suggested by J. Murakami. The proof is left to the reader.

PROPOSITION A.10.

$$\left\langle \begin{array}{c} i \uparrow \\ j+k \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{j+k-i} \\ \xleftarrow{k} \end{array} \begin{array}{c} j+l \uparrow \\ i-k+l \uparrow \\ i+l \uparrow \end{array} \right\rangle_n = \sum_{m=0}^i \begin{bmatrix} l \\ k-m \end{bmatrix} \left\langle \begin{array}{c} i \uparrow \\ i-m \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{m} \\ \xrightarrow{j+m-i} \end{array} \begin{array}{c} j+l \uparrow \\ j+m+l \uparrow \\ i+l \uparrow \end{array} \right\rangle_n$$

for  $j \geq i \geq k \geq 1$ . Here  $\begin{bmatrix} x \\ y \end{bmatrix} = 0$  if  $y < 0$  or  $y > x$ .