4. SOME EXAMPLES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

4. Some examples

We give here three simple examples of the type of bigraded ring which might result from this construction. Each of these examples is obtained by taking a birational map from a regular scheme to $\mathrm{Spec}(R/\mathfrak{q})$, and the last two are simple resolutions of singularities.

We first summarize the construction up to this point. We began with a regular local ring R and prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. We then took a regular subscheme Z' of $\mathrm{Proj}(R[X_0, \dots, X_n])$ which was generically finite over $\mathrm{Spec}(R/\mathfrak{q})$. The next step was to replace $R[X_0, \dots, X_n]$ with the associated graded ring of I tensored with $R/\mathfrak{m} = k$. The sheaf $\mathcal{O}_{Y'}$ defined by $B = R/\mathfrak{p}[X_0, \dots, X_n]$ was then replaced with the sheaf $\mathcal{M}$ defined by $gr_I(B)$, again tensored with k. The assumption of regularity implies that I/I^2 is locally free over A/I; denote its rank r. Then the dimension of $\mathcal{M}$ is at most r, and it is equal to r if and only if we had $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. We note that the fiber Z'_s of Z' over the maximal ideal of R has dimension at most $\dim(R/\mathfrak{q}) - 1$, but apart from that we do not know much about it. It is the projective scheme defined by the graded ring $(A/I) \otimes_R k$, which is the part of degree zero in the grading in the second component of the bigraded ring we are considering.

For the first example, let R have dimension four, let t, u, v, w be a regular system of parameters, and define the prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ by letting $\mathfrak{p} = (t, u)$ and $\mathfrak{q} = (v, w)$. In this case, $\mathrm{Spec}(R/\mathfrak{q})$ is already regular, and we can simply take the projective scheme $\mathrm{Proj}(R[X]) = \mathrm{Spec}(R)$.

For a slightly more complicated example, consider the subscheme of the projective space $\mathrm{Proj}(R[X, Y])$ defined by the ideal I generated by v, w, and $Xu - Yt$. Then Z' is the blow up of $R/\mathfrak{q}$ at the point defined by the maximal ideal, and the fiber over s is projective space of dimension 1. One could define similar examples in higher dimension.

For a third example, let R have dimension 2, and let I be generated by $Xu - Yt, Zu - Xt, X^2 - YZ$. The projective space P has dimension 2, and the fiber over the maximal ideal has codimension one in $\mathrm{Proj}(k[X, Y, Z])$ and thus has dimension one. The sheaf defined by I/I^2 has rank 2, but I is minimally generated by three elements.

In the above examples it was not really necessary to reduce to projective space since the original quotients $R/\mathfrak{q}$ were regular. We next consider an example where the original scheme is not regular. Let $\mathfrak{m}$ be minimally generated by t, u, and let $\mathfrak{q}$ be the principal ideal generated by $t^2 - u^3$.

We can resolve the singularity by letting Z' be defined by the ideal in $R[X, Y]$ generated by $t^2 - u^3$, $uX - tY$, $X^2 - uY^2$. The fiber Z'_s in this case is $\mathrm{Proj}(k[X, Y]/(X^2))$.

Finally, we consider the case where $\mathfrak{q}$ is the determinantal ideal in R of dimension 4 generated by $wu - t^2$, $wv - tu$, and $tv - u^2$. In this case the resolution can be found by taking the ideal I in $R[X, Y, Z, W]$ generated by the following elements:

$$Z^2 - YW, YZ - XW, Y^2 - XZ, uW - vZ, uZ - vY, uY - vX, u^2 - tv,$$
$$tW - vY, tZ - vX, tY - uX, tu - wv, t^2 - wu, wW - vX, wZ - uX, wY - tX.$$

The fiber over the maximal ideal is a determinantal subvariety of dimension 1.

In a later section we will return to these examples and consider the question of computing the Euler characteristics $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$ for sheaves $\mathcal{M}$ defined as above by certain prime ideals $\mathfrak{p}$ of R.

5. Hilbert polynomials of bigraded modules

In section 2 we showed how the Serre spectral sequence can be used to express the Euler characteristic defined by a Koszul complex in terms of the Samuel multiplicity. In this section we show that similar results hold in the present situation. We now let C denote the bigraded ring which we previously denoted $gr_I(A) \otimes_R k$, where $C_{i,j}$ consists of the elements of $(I^j/I^{j+1}) \otimes k$ of degree i. Thus in our present notation, $E_s = \mathrm{Proj}(C)$, where the grading on C is that in the first coordinate. Let C_0 denote the subring $\bigoplus_i (C_{i,0})$. Let r be the rank of I/I^2, and let M be a bigraded module defining a sheaf $\mathcal{M}$ on E_s of dimension at most r; we define the dimension of M to be the dimension of the associated sheaf. We consider the question of computing the Euler characteristic $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$, which we also denote $\chi(C_0, M)$.

Let

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to C_0 \to 0$$

be a complex of bigraded modules which defines a locally free resolution of C_0 over C. For any finitely generated bigraded module N, we let $P_N(m, n)$ by the Hilbert polynomial of N; more precisely, we define P_N to be the polynomial in two variables such that

$$P_N(m, n) = \sum_{i=0}^{n-1} \mathrm{length}(N_{m,i})$$