8.2 The case of multiple roots

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

Clearly $v > 0$; moreover

$$u = \frac{\dfrac{p_{h-1}}{q_{h-1}} - \dfrac{q_h}{q_{h-1}} \dfrac{p_h}{q_h}}{1 - \dfrac{q_h}{q_{h-1}}} = \frac{p_{h-1} - p_h}{q_{h-1} - q_h} > 0\,.$$

It follows that this circle is entirely contained in the right half plane. A root x_j different from x_0 (see Fig. 3) lies outside the circle (8.3) if h is large enough to have $\Delta > |b - a|$. It follows that $\operatorname{Re}\mathcal{F}(x_j) < 0$, and hence it is external to the circle (8.6) corresponding to the value $h+1$. Hence the condition

$$F_{h-1} F_{h-2} \Delta > 1$$

ensures that the polynomial f_{h+1} is reduced.

8.2 THE CASE OF MULTIPLE ROOTS

Obreschkoff's Lemma 8.1 yields the following

COROLLARY 8.3. *Let* $f(x) = (x - x_1)(x - x_2) \cdot \ldots \cdot (x - x_r)$, *where* $x_i \in \mathbf{R}^+$. *Then*

$$f_1(x) = (x^2 + 2\rho x \cos\varphi + \rho^2) f(x)\,, \quad \rho > 0\,, \quad |\varphi| < \frac{\pi}{r+2}$$

has exactly r *variations. More generally, a polynomial having* r *positive real roots and all its other roots in the sector*

$$S = \left\{ x \,\middle|\, x = -\rho(\cos\varphi + i\sin\varphi), \quad \rho > 0, \quad |\varphi| < \frac{\pi}{r+2} \right\}$$

has exactly r *variations.*

This allows us to extend Vincent's theorem to the case of multiple roots. Suppose the polynomial $f(x)$ has multiple roots, and let Δ be their least distance. If h is sufficiently large to verify

$$F_h F_{h-1} \Delta > 1\,,$$

at most one root x_0 lies in (a, b), but since this root may have multiplicity r, f_h has 0 or at least r variations. It will have exactly r variations if we can ensure that $x_0 \in (a, b)$ and that the other transformed roots lie in the sector

$$S = \{ y \mid \operatorname{Re} y < 0, \quad |\operatorname{Im} y| < |\tan\varphi| \cdot |\operatorname{Re} x| \}\,,$$

where $\varphi = \dfrac{\pi}{r+2}$. Let $s = \tan\dfrac{\pi}{r+2}$ and let us make the appropriate substitutions into (8.4). We have proved

THEOREM 8.4. *Let $f(x)$ be a real polynomial of degree n whose roots are of multiplicity smaller than r. Let $\gamma = [c_0, c_1, c_2, \dots]$ and, maintaining the previous notation, consider the polynomials*

$$f_{h+1} = (q_{h-1} + q_h x) f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right).$$

Let $s = \tan \frac{\pi}{r+2}$. If h satisfies

$$F_h F_{h-1} \Delta > \sqrt{1 + \frac{1}{s^2}} = \frac{1}{\sin \frac{\pi}{r+2}},$$

then the number of variations of f_{h+1} equals the multiplicity of the root in $\left(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\right)$.

REMARK 9. Obviously, letting $s = \tan \frac{\pi}{n+2}$, we can implement an algorithm to isolate the roots, without being forced to reduce the polynomial $f(x)$ to one with simple roots.

REMARK 10. We conclude our paper by showing that our estimate of the size of h is asymptotically better than Chen's. Suppose we consider a polynomial whose roots are of multiplicity $\leq r$ (which necessarily has degree $n \geq r$). We have proved that the isolation of a root can be carried out in p steps, where p verifies

$$F_p F_{p-1} \Delta > \sqrt{1 + \frac{1}{\tan^2 \frac{\pi}{r+2}}} = \frac{1}{\sin \frac{\pi}{r+2}}. \tag{8.7}$$

We want to compare this integer with that needed by Chen's theorem, that is the smallest integer $m = h + k$, where h and k satisfy

$$F_h F_{h-1} \Delta > 1 \quad \text{and} \quad k > \tfrac{1}{2} \log_\phi r. \tag{8.8}$$

We know that

$$\sqrt{5} F_k \approx \phi^{k+1},$$

hence (8.7) becomes

$$\frac{\Delta}{5} \phi^{2p+1} > \frac{1}{\sin \frac{\pi}{r+2}}.$$

On the other hand, by (8.8) we have

$$\frac{\Delta}{5} \phi^{2h+1} > 1 \quad \text{and} \quad k > \tfrac{1}{2} \log_\phi r. \tag{8.9}$$

The second equality may be rewritten as

$$\phi^{2k} > r. \tag{8.10}$$

From the first inequality of (8.9) and (8.10) it follows that

$$\frac{\Delta}{5}\phi^{2h+1}\phi^{2k} = \frac{\Delta}{5}\phi^{2(h+k)+1} > r.$$

Hence

$$\frac{\Delta}{5}\phi^{2m+1} > r.$$

Since

$$r > \frac{1}{\sin\frac{\pi}{r+2}} \quad \text{for } r \geq 2,$$

$m \geq p$ for r sufficiently large and the proof is concluded. □

ACKNOWLEDGMENTS. We are indebted to Xu Kang, Alberto Setti and Giancarlo Travaglini, for the help they gave us in preparing this paper.

REFERENCES

[1] AKRITAS, A. G. Vincent's theorem in algebraic manipulation. Ph. D. Thesis, Operation Research Program, North Carolina State University, Raleigh, N.C., 1978.

[2] —— A new method for polynomial real root isolation[20]. *Proceedings of the 16th annual southeast regional ACM conference*, Atlanta, Georgia, April 1978, 39–43.

[3] —— A correction on a theorem by Uspensky. *Bull. Soc. Math. Grèce (N.S.) 19* (1978), 278–285.

[4] —— Reflections on a pair of theorems by Budan and Fourier. *Math. Mag. 55* (1982), 292–298.

[5] —— There is no "Uspensky's method". Extended abstract. *Proceedings of the 1986 Symposium on Symbolic and Algebraic Computation,* Waterloo, Ontario, Canada, 1986, 88–90.

[6] —— The role of the Fibonacci sequence in the isolation of the real roots of polynomial equations. *Applications of Fibonacci Numbers,* vol. 3, 1988.

[7] —— *Elements of Computer Algebra with Applications.* John Wiley & Sons, New York, 1989.

[8] AKRITAS, A. G. and S. D. DANIELOPOULOS. On the forgotten theorem of Mr. Vincent. *Historia Math. 5* (1978), 427–435.

[9] AKRITAS, A. G. and S. D. DANIELOPOULOS. A converse rule of signs for polynomials. *Computing 34* (1985), 283–286.

[20]) This paper won the first prize in the student paper competition.