§0. Introduction and preliminaries

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§0. INTRODUCTION AND PRELIMINARIES

Let R be a commutative ring and let $q \in R$ be an invertible element. For any integer $n \geq 1$ let $H_n(q)$ be the Hecke algebra of type A_{n-1}, with standard generators $T_1, \dots, T_{n-1}$ and let $H_n^a(q) \supset H_n(q)$ be the corresponding affine Hecke algebra. Thus $H_n^a(q)$ has generators $T_1, \dots, T_n$ which satisfy

$$\text{(0.1a)} \qquad \begin{aligned} T_iT_j &= T_jT_i && \text{if } |i-j| \geq 2 \text{ and } \{i,j\} \neq \{1,n\}, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && i = 1, \dots, n, \text{ the subscripts being taken mod } n. \end{aligned}$$

and

$$\text{(0.1b)} \qquad (T_i - q)(T_i + q^{-1}) = 0\,.$$

It is well known that $H_n^a(q)$ has an R-basis consisting of elements T_w, where w runs over the elements of the affine Weyl group W^a of type $\widetilde{A}_{n-1}$ and that $H_n(q)$ is the free R-submodule with basis $\{T_w \mid w \in W\}$, where W is the Weyl group of type A_{n-1}. The T_w are defined as follows. Let $r_1, \dots, r_n$ be the set of simple reflections for W^a which correspond to the T_i. If $\ell(w)$ is the corresponding length function on W^a, for any $w \in W^a$ take a reduced expression $w = r_{i_1} \dots r_{i_\ell}$ $(\ell = \ell(w))$; then $T_w = T_{i_1} \dots T_{i_\ell}$.

For each $i = 1, \dots, n$ the group $W(i)$ generated by r_i and r_{i+1} is isomorphic to the symmetric group $Sym(3)$ (where i is taken modulo n). Thus we may form the quasi-idempotents

$$\text{(0.2)} \qquad E_i = \sum_{w \in W(i)} q^{\ell(w)} T_w$$

for $i = 1, \dots, n$. These satisfy

$$\text{(0.3)} \qquad E_i^2 = \Big(\sum_{w \in W(i)} q^{2\ell(w)}\Big)E_i = (1-q^2)(1-q^4)E_i\,.$$

Let $I_n^a = (E_1, \dots, E_n)$ be the ideal of $H_n^a(q)$ which is generated by $E_1, \dots, E_n$. It is well known (cf. e.g. [J1]) that the Temperley-Lieb algebra $\mathbf{T}(n) = TL_n(\delta)$ (where $\delta = -(q+q^{-1})$) may be defined by

$$\text{(0.4)} \qquad TL_n = H_n(q)/I_n$$

where $I_n = I_n^a \cap H_n(q)$ is the ideal of $H_n(q)$ generated by $E_1, \dots, E_{n-1}$.

Its affine analogue is $TL_n^a(\delta)$, defined by

$$\text{(0.5)} \qquad TL_n^a(\delta) = H_n^a(q)/I_n^a\,.$$

In this work we shall study algebras $\mathbf{T}^a(n)$ which are slightly larger than TL_n^a (see (2.9) below). They are obtained from TL_n^a by adding a "twist" (denoted τ below). It is these algebras (the $\mathbf{T}^a(n)$) which we refer to as the *affine Temperley-Lieb algebras*. We shall define the algebras in terms of what we shall call the *Temperley-Lieb category* $\mathbf{T}^a$ (see §2 below), whose objects are the non-negative integers $\mathbf{Z}_{\geq 0}$ and whose morphisms $\mathbf{T}^a(n, m)$ $(n, m \in \mathbf{Z}_{\geq 0})$ are R-linear combinations of "affine diagrams" from n to m (see (1.3) below). The algebra $\mathbf{T}^a(n)$ is then just $\mathbf{T}^a(n, n)$. We shall show (in (2.9) below) that the algebras defined in this way contain the Temperley-Lieb quotients TL_n^a of the affine Hecke algebra (cf. [Ch] or [Lu2]).

If W is any functor from $\mathbf{T}^a$ to the category R-**mod** of R-modules, then $W(n)$ is a $\mathbf{T}^a(n)$-module (for $n \in \mathbf{Z}_{\geq 0}$). We shall construct such functors $W_{t,z}$ for each pair (t, z) such that $t \in \mathbf{Z}_{\geq 0}$ and z is an invertible element of R. For each such pair (t, z) we shall define an invariant bilinear form

$$\langle\ .\ \rangle_{t,z} \colon W_{t,z} \times W_{t,z^{-1}} \to R$$

which is "generically" non-degenerate. When R is a field, we show that if $\mathrm{rad}_{t,z}$ is the radical of $\langle\ .\ \rangle_{t,z}$, then $L_{t,z} := W_{t,z}/\mathrm{rad}_{t,z}$ is absolutely irreducible and all irreducible finite dimensional $\mathbf{T}^a(n)$-modules are of this form.

We then give a characterisation of all homomorphisms between the cell modules (Theorem (3.4) below). This is in some sense the main result of this paper, as it enables us to determine the decomposition matrices of the cell modules. Section 4 is concerned with the determination of the discriminants of the forms $\langle\ .\ \rangle_{t,z}$. This gives explicit results concerning the semisimplicity of related finite dimensional algebras. In §5 we give the composition multiplicities of the components of the cell modules and derive corresponding statements for the related finite dimensional algebras. In (3.7) we derive, as a by-product of our explicit determination of the homomorphisms between the cell modules, a closed formula for the Jones or augmentation idempotent (see [MV], [We] and [Li]) of the Temperley-Lieb algebra when q is a root of unity. In [MV] certain coefficients of this idempotent are computed, while in [We] a recursive formula is given for it. Our formula (see (3.7) below) differs from these by being explicit and closed, although it only applies when q is a root of unity. It leads to a presentation of Jones' projection algebra when the Jones trace on the Temperley-Lieb algebra is not non-degenerate. This includes those values of q for which the Temperley-Lieb algebra is not semisimple. Since our algebras contain quotients of the affine Hecke algebras of type A, their representations yield representations of the affine Hecke algebras. Hence our results may be interpreted as a contribution to the representation theory of these affine Hecke algebras at roots of unity (see [KL1, KL2]). Our construction of representations of the algebras through functors on the category of diagrams may share some ideas with [FY] or [RT], although we have no heuristic explanation for the fact that all irreducible representations arise functorially (Theorem (2.8) below).

TOTALLY ORDERED SETS. In order to define the diagrams below, we introduce some constructions associated with totally ordered sets.

(0.6) If X and Y are totally ordered sets, form a new totally ordered set $X\#Y$ as follows. The underlying set is the disjoint union $X \amalg Y$ of X and Y. Let $\ell\colon X \to X \amalg Y$ and $u\colon Y \to X \amalg Y$ denote the canonical injections. Define the total order by stipulating that (in increasing order) the elements of (the image) $\ell(X)$ come first in reverse order, followed by the elements of $u(Y)$ in natural order. Intuitively, $X\#Y$ should be imagined as two horizontal lines in the real affine plane with the elements of X (identified with $\ell(X)$) on the lower line and those of Y (identified with $u(Y)$) on the upper line. The ordering is given by moving leftwards along the bottom line, then right along the top. An element of $X\#Y$ is said to be *lower* (or a lower vertex) if it lies in $\ell(X)$ and *upper* (or an upper vertex) if it lies in $u(Y)$.

(0.7) Given any totally ordered set X, form a new totally ordered set with a distinguished automorphism as follows. Denote by $\mathbf{Z} \times X$ the set of pairs (i,x) where i is an integer and $x \in X$ and order this set lexicographically: $(i_1,x_1) < (i_2,x_2)$ if $i_1 < i_2$ or $i_1 = i_2$ and $x_1 < x_2$. Define the automorphism V_X of $\mathbf{Z} \times X$ by $(i,x) \mapsto (i+1,x)$ $(i \in \mathbf{Z}, x \in X)$. If Y is another totally ordered set, then we may extend the permutations V_X and V_Y to the automorphism $V_X\#V_Y$ of $(\mathbf{Z} \times X)\#(\mathbf{Z} \times Y)$, given by $\ell(i,x) \mapsto \ell(i+1,x)$ for i in $\mathbf{Z}$ and x in X and $u(i,y) \mapsto u(i+1,y)$ for i in $\mathbf{Z}$ and y in Y. When there is no danger of confusion, we shall abbreviate V_X and $V_X\#V_Y$ to V.

We shall require the following result.

(0.8) LEMMA (cf. [GL, (4.5)]). *Let V be a permutation of a set X and assume that X has finitely many V orbits. Let ϕ_1 and ϕ_2 be involutory permutations of X which commute with V. Assume that the fixed point sets* $\mathrm{fix}(\phi_1)$ *and* $\mathrm{fix}(\phi_2)$ *are disjoint. Then the orbits $\mathcal{O}$ of the subgroup H (of permutations of X) generated by ϕ_1 and ϕ_2, fall into the following mutually exclusive classes:*

(1) *$\mathcal{O}$ contains no points in* $\mathrm{fix}(\phi_1) \cup \mathrm{fix}(\phi_2)$ *; in this case we call $\mathcal{O}$ a* loop.

(2) *$\mathcal{O}$ contains exactly two points in* $\mathrm{fix}(\phi_1) \cup \mathrm{fix}(\phi_2)$ *; in this case we call $\mathcal{O}$ an* arc *and refer to the two fixed points as the* ends *of the arc.*

When the ends of an arc (case (2) above) are not in the same set $\mathrm{fix}(\phi_i)$ ($i = 1$ or 2) we say the orbit is a *through arc.*

Proof. Suppose that an orbit $\mathcal{O}$ contains a point x of $\mathrm{fix}(\phi_1)$ (say). Following [GL, (4.5)], write $(\phi_1\phi_2)_i = \dots\phi_2\phi_1\phi_2$ (i factors) and write $x_i = (\phi_1\phi_2)_i x$ (for $i = 0, 1, 2, \dots$), so that $x_0 = x$ etc. Then clearly $\mathcal{O} = \{x_0, x_1, \dots\}$. If the orbit $\mathcal{O}$ is finite, the result is immediate by the argument in [GL, loc. cit.]. If $\mathcal{O}$ is infinite, then two of its elements lie in the same V-orbit by finiteness, whence there are indices $i < j$ and $k \in \mathbf{Z}$ such that $V^k x_i = x_j$. Acting by ϕ_1 and ϕ_2, it follows that $V^k x_0 = x_{j \pm i} = x_r$ for some $r > 0$. Hence x_r is fixed by ϕ_1. It follows, using the same argument as in [GL, loc. cit.] that $\mathcal{O} = \{x_0, \dots, x_r\}$, which contradicts the infinite nature of $\mathcal{O}$. $\square$

Notice that the proof of (0.8) shows that any infinite H-orbits must be loops. Also, if X is finite, V may (and generally will) be trivial.

§ 1. INVOLUTIONS, DIAGRAMS AND CATEGORIES

We shall consider various categories in this work whose objects are the non-negative integers $\mathbf{Z}_{\geq 0}$. The morphisms in these categories are defined in terms of "diagrams" and their "composition", whose definition in turn depends on the notion of a "planar involution" (cf. [GL, § 6]). In this section we develop a calculus of involutions and diagrams; our principal purpose is the definition of the category $\mathbf{D}^a$ of affine diagrams. These generalise the familiar diagrams which may be used to define the ordinary Temperley-Lieb algebra $\mathbf{T}(n)$.

(1.1) DEFINITION.

(1) A *planar involution* of the totally ordered set P is a permutation ϕ of P such that ϕ^2 is the identity, ϕ has no fixed points and if $x, y \in P$ then $x \leq y \leq \phi(x) \Rightarrow x \leq \phi(y) \leq \phi(x)$.

(2) If t and n are non-negative integers, a *finite diagram* $\alpha\colon t \to n$ is a planar involution ϕ_α of $\mathbf{t}\#\mathbf{n}$, where the latter set is defined in (0.6).