5. $0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0$ WITH $\bar{E}$ FLAT

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **18.09.2024**

5. $0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$ WITH $\bar{E}$ FLAT

Throughout this section we will assume that the hermitian vector bundle $\bar{E}$ is *flat*, i.e. that $K_E = 0$. As before, the metrics h_S and h_Q will be induced from the metric on E. Define the *harmonic numbers* $\mathcal{H}_k = \sum_{i=1}^{k} \frac{1}{i}$, $\mathcal{H}_0 = 0$.

Let λ be a partition of k (we denote this by $\lambda \vdash k$). Recall that the polynomials $\{p_\lambda : \lambda \vdash k\}$ form a $\mathbf{Q}$-basis for the vector space of symmetric homogeneous polynomials in $x_1, \dots, x_n$ of degree k. The following result computes the Bott-Chern form corresponding to any such invariant polynomial:

THEOREM 3. *The Bott-Chern class $\widetilde{p_\lambda}(\overline{\mathcal{E}})$ in $\widetilde{A}(X)$ is the class of*

(i) $k\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$, *if* $\lambda = k = (k, 0, 0, \dots, 0)$

(ii) 0, *otherwise.*

Proof. Let us first compute $\widetilde{p_k}(\overline{\mathcal{E}})$ for $p_k(A) = \mathrm{Tr}(A^k)$. Since $K_E = 0$, the deformed matrix $K(u) = (1-u)K_{S\oplus Q}$, where $K_{S\oplus Q} = \left(\begin{array}{c|c} K_S & 0 \\ \hline 0 & K_Q \end{array}\right)$. Since

$$\int_0^1 \frac{(1-u)^{k-1}-1}{u}\, du = -\int_0^1 \frac{t^{k-1}-1}{t-1}\, dt = -\mathcal{H}_{k-1},$$

we obtain

$$\widetilde{p_k}(\overline{\mathcal{E}}) = -\mathcal{H}_{k-1}p_k'(K_{S\oplus Q}; J_r) = -k\mathcal{H}_{k-1}\,\mathrm{Tr}(K_S^{k-1}) = -k\mathcal{H}_{k-1}p_{k-1}(\overline{S}).$$

Now since $p_k(\bar{S} \bigoplus \bar{Q}) - p_k(\bar{E})$ is exact, $p_k(\bar{E}) = 0$ and $p_k(\bar{S} \bigoplus \bar{Q}) = p_k(\bar{S}) + p_k(\overline{Q})$, we conclude that $p_k(\bar{S}) = -p_k(\overline{Q})$ in $\widetilde{A}(X)$, for each $k \geqslant 1$. This proves (i).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a partition ($m \geqslant 2$). Proposition 1 implies that

$$\widetilde{p_\lambda}(\overline{\mathcal{E}}) = \widetilde{p_{\lambda_1}}(\overline{\mathcal{E}})p_{\lambda_2}\cdots p_{\lambda_m}(\bar{S} \bigoplus \bar{Q}).$$

But $\widetilde{p_{\lambda_1}}(\overline{\mathcal{E}})$ is a closed form (by (i)), and $p_{\lambda_2}\cdots p_{\lambda_m}(\bar{S} \bigoplus \bar{Q})$ is an exact form. Thus $\widetilde{p_\lambda}(\overline{\mathcal{E}})$ is exact, and so vanishes in $\widetilde{A}(X)$. □

It follows from Theorem 3 that for any $\phi \in I(n)$, the Bott-Chern form $\widetilde{\phi}(\overline{\mathcal{E}})$ is a linear combination of homogeneous components of the Chern character form $ch(\overline{Q})$. In [Ma], Theorem 3.4.1 we find the calculation

(4) $$\widetilde{c_k}(\overline{\mathcal{E}}) = \mathcal{H}_{k-1}\sum_{i=0}^{k-1} i\, c_i(\bar{S})c_{k-1-i}(\bar{Q})$$

for the Chern forms $\widetilde{c_k}$. Our result gives the following

PROPOSITION 3. $\widetilde{c_k}(\overline{\mathcal{E}}) = (-1)^{k-1}\mathcal{H}_{k-1}p_{k-1}(\bar{Q})$.

Proof. By Newton's identity (2) we have

(5) $$\widetilde{p_k} - \widetilde{c_1p_{k-1}} + \widetilde{c_2p_{k-2}} - \cdots + (-1)^k k\widetilde{c_k} = 0\,.$$

Reasoning as in Theorem 3, we see that if ϕ and ψ are two homogeneous invariant polynomials of positive degree, then $\widetilde{\phi\psi}(\overline{\mathcal{E}}) = 0$ in $\widetilde{A}(X)$. Thus (5) gives $\widetilde{c_k}(\overline{\mathcal{E}}) = \dfrac{(-1)^{k-1}}{k}\widetilde{p_k}(\overline{\mathcal{E}}) = (-1)^{k-1}\mathcal{H}_{k-1}p_{k-1}(\bar{Q})$. □

REMARK. The result of Proposition 3 agrees with (4), i.e. $(-1)^k p_k(\bar{Q}) = \sum_{i=0}^{k} i\, c_i(\bar{S})c_{k-i}(\bar{Q})$ in $\widetilde{A}(X)$. To see this, let $h(t) = \sum c_i(\bar{S})t^i$, $g(t) = \sum c_j(\bar{Q})t^j$, and $f(t) = \sum i\, c_i(\bar{S})t^i$. Then $h(t)g(t) = 1$ in $\widetilde{A}(X)[t]$, and $f(t) = th'(t)$. Choose formal variables $\{x_\alpha\}_{1\leqslant\alpha\leqslant r}$ and set $c_i(\bar{S}) = e_i(x_1,\ldots,x_r)$, so that $h(t) = \prod_\alpha(1+x_\alpha t)$. Then $f(t) = \sum_\alpha tx_\alpha \prod_{\beta\neq\alpha}(1+x_\beta t)$. Thus

$$\begin{aligned} f(t)g(t) = \frac{f(t)}{h(t)} &= \sum_\alpha \frac{x_\alpha t}{1+x_\alpha t} \\ &= r - \sum_\alpha \frac{1}{1+x_\alpha t} = r - \sum_{\alpha,i}(-1)^i x_\alpha^i t^i = r - \sum_i (-1)^i p_i(\bar{S})t^i \\ &= r + \sum_i (-1)^i p_i(\bar{Q})t^i\,. \end{aligned}$$

Comparing coefficients of t^k on both sides gives the result.

We can use Theorem 3 to calculate $\widetilde{\phi}(\overline{\mathcal{E}})$ for $\phi \in I(n)_k$: it is enough to find the coefficient of the power sum p_k when ϕ is expressed as a linear combination of $\{p_\lambda\}_{\lambda\vdash k}$ in $\Lambda(n,\mathbf{Q})$. For example, we have

COROLLARY 2. $\widetilde{ch}(\overline{\mathcal{E}}) = \sum_k \mathcal{H}_k ch_k(\bar{Q})$, *where* ch_k *denotes the* k*-th homogeneous component of the Chern character form.*

COROLLARY 3. *Let λ be a partition of k and s_λ the corresponding Schur polynomial in $\Lambda(n,\mathbf{Q})$. Then $\widetilde{s_\lambda}(\overline{\mathcal{E}})=0$ unless λ is a hook $\lambda^i=(i,1,1,\dots,1)$, in which case $\widetilde{s_{\lambda^i}}(\overline{\mathcal{E}})=(-1)^{k-i}\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$.*

Proof. The proof is based on the Frobenius formula

$$s_\lambda=\frac{1}{k!}\sum_{\sigma\in S_k}\chi_\lambda(\sigma)p_{(\sigma)}$$

where (σ) denotes the partition of k determined by the cycle structure of σ (cf. [M], §I.7). By the above remark, $\widetilde{s_\lambda}(\overline{\mathcal{E}})=\chi_\lambda\big((12\dots k)\big)\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$. Using the combinatorial rule for computing χ_λ found in [M], p. 117, Example 5, we obtain

$$\chi_\lambda((12\dots k))=\begin{cases}(-1)^{k-i}, & \text{if } \lambda=\lambda^i \text{ is a hook}\\ 0, & \text{otherwise}.\end{cases}$$

□

The most natural instance of a sequence $\overline{\mathcal{E}}$ with $\overline{E}$ flat is the classifying sequence over the Grassmannian $G(r,n)$. As we shall see in §8, the calculation of Bott-Chern forms for this sequence leads to a presentation of the Arakelov Chow ring of the arithmetic Grassmannian over $\operatorname{Spec}\mathbf{Z}$.

6. CALCULATIONS WHEN $\overline{E}$ IS PROJECTIVELY FLAT

We will now generalize the results of the last section to the case where E is *projectively flat*, i.e. the curvature matrix K_E of $\overline{E}$ is a multiple of the identity matrix: $K_E=\omega\, Id_n$. This is true if $E=\overline{L}^{\oplus n}$ for some hermitian line bundle $\overline{L}$, with $\omega=c_1(\overline{L})$ the first Chern form of $\overline{L}$.

The Bott-Chern forms (for the induced metrics) are always closed in this case as well, and will be expressed in terms of characteristic classes of the bundles involved. However this seems to be the most general case where this phenomenon occurs.

The key observation is that for projectively flat bundles, the curvature matrix $K_E=\omega Id_n$ in *any* local trivialization. Thus we have

$$K(u)=\left(\begin{array}{c|c}(1-u)K_S+u\,\omega\, Id_r & 0\\ \hline 0 & (1-u)K_Q+u\,\omega\, Id_s\end{array}\right)$$

where $s=n-r$ denotes the rank of Q. Now Theorem 2 gives