

## 6. WHAT ABOUT q?

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

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6. WHAT ABOUT  $q$  ?

We pose the question: is  $q$  necessarily a  $q^s$ th power modulo  $p$  in Theorem 2? A numerical test will quickly show that this is most certainly not always so. However, Theorem 3 tells us that the answer is in fact “yes” in most cases.

*Proof of Theorem 3.* Let  $K, L, \pi$  be as in the proof of Theorem 2 and assume  $s > 0$ . Note first that  $(q/\pi)_l = (\pi, q)_l$  by reciprocity law (6). We will evaluate the latter symbol.

Viewing  $\Phi_{n/l}(X, Y)$  as a polynomial over  $K$ , we have that

$$\pi = N_{L/K}(y - qx\zeta_n) = \Phi_{n/l}(y, qx\zeta_l),$$

where  $\zeta_l$  in this equation is given by  $\zeta_l^{n/l} = \zeta_n^{n/l}$ . We now state the following generating formula for homogeneous cyclotomic polynomials:

$$(8) \quad \Phi_m(X, Y) = \frac{X^m - Y^m}{\prod_{\substack{d|m \\ 0 < d < m}} \Phi_d(X, Y)}.$$

Applying this formula recursively, we see that  $\pi$  is expressible as a product of numbers of the form  $y^r - (qx\zeta_l)^r$  and reciprocals of such numbers, where  $r$  is some positive divisor of  $n/l$ . To show that  $(\pi, q)_l = 1$ , it is by bimultiplicativity enough to show that  $(y^r - (qx\zeta_l)^r, q)_l = 1$  for all such  $r$ . And since  $n/l$  is relatively prime to  $q$ , it will clearly suffice to show that  $(y - qx\zeta_l, q)_l = 1$  for any choice of  $\zeta_l$  and integers  $x$  and  $y$  with  $y$  relatively prime to  $q$ .

We have

$$(y - qx\zeta_l, q)_l = (y, q)_l (1 - qxy^{-1}\zeta_l, q)_l.$$

The first symbol  $(y, q)_l$  is fixed under the action of the Galois group  $G_{\mathbf{Q}_q(\zeta_l)/\mathbf{Q}_q}$  by Theorem 6(h) since  $y, q \in \mathbf{Q}_q$ . As an  $l$ th root of unity with  $l$  odd, it must therefore be 1.

By Theorem 6(f),  $(1 - qxy^{-1}\zeta_l, qxy^{-1}\zeta_l)_l = 1$ . But by bimultiplicativity, this means that

$$(1 - qxy^{-1}\zeta_l, q)_l = (1 - qxy^{-1}\zeta_l, xy^{-1})_l^{-1} (1 - qxy^{-1}\zeta_l, \zeta_l)_l^{-1}.$$

Corollary 8 yields that  $1 - qxy^{-1}\zeta_l \equiv 1 \pmod{f_l(xy^{-1})}$ , and so the first symbol on the right is 1. The second symbol can be evaluated by turning it back into a power residue symbol and applying (4). Since  $\zeta_l$  is a unit in the ring of integers of  $K$ , the reciprocity law (5) yields

$$(9) \quad (1 - qxy^{-1}\zeta_l, \zeta_l)_l = \left( \frac{\zeta_l}{1 - qxy^{-1}\zeta_l} \right)_l = \zeta_l^{(N_K(1 - qxy^{-1}\zeta_l) - 1)/l}.$$

Thus  $(1 - qxy^{-1}\zeta_l, \zeta_l)_l$  will equal 1 if and only if  $N_K(1 - qxy^{-1}\zeta_l) \equiv 1 \pmod{q^{2s}}$ . In fact,

$$N_K(1 - qxy^{-1}\zeta_l) = \sum_{i=0}^{q-1} (qxy^{-1})^{iq^{s-1}} \equiv 1 \pmod{q^{q^{s-1}}}.$$

It is easily seen that  $q^{s-1} \geq 2s$  exactly when stated in the theorem.  $\square$

One remark on the case  $s = 1$ . If in fact we take  $n = q$ , then since  $\Phi_q(X) = 1 + X + \dots + X^{q-1}$  we have that  $p \equiv 1 \pmod{q^2}$  if and only if  $q$  divides  $x$ . Then  $q$  is a  $q$ th power modulo  $p$  if and only if  $x$  is divisible by  $q$ , in stark contrast to the above theorem.

## 7. THE EVEN CASE

We now turn to the case of  $q = 2$ . Given a positive integer  $s$ , let us set  $l = 2^s$ . We refrain from proving the theorem for the more general case of homogeneous polynomials, though it holds under such a generalization.

Any  $\alpha \in \mathbf{Q}_2^*$  may be written uniquely as  $\alpha = \xi 2^b (-3)^c$  where  $\xi = \pm 1$ ,  $b \in \mathbf{Z}$ , and  $c \in \mathbf{Z}_2$ . Note that  $b = v_2(\alpha)$ , where  $v_2$  is the 2-adic valuation. Denote by  $f_l(\alpha)$  the conductor of the norm residue character  $(\cdot, \alpha)_l$  in  $\mathbf{Q}_2(\zeta_l)$ . The conductors in this case have been worked out by Despina Prapavessi in [P]. We use a corrected version of her theorem [Sh1].

**THEOREM 9 (Prapavessi).** *Let  $\alpha \in \mathbf{Q}_2^*$  and write  $\alpha = \xi 2^b (-3)^c$  as above. Let  $w = \min \{v_2(b), v_2(c) + 2\}$ . Then if  $\xi = 1$ ,*

$$f_l(\alpha) = \begin{cases} (8) & \text{if } w = 0, \\ (4) & \text{if } w = 1 \text{ and } s \geq 2, \\ (\lambda_{2^{w-1}}) & \text{if } 2 \leq w \leq s \text{ and } w = v_2(c) + 2, \\ (\lambda_{2^w} \lambda_{2^{w+1}}) & \text{if } 2 \leq w < s - 1 \text{ and } w \leq v_2(c) + 1, \\ (\lambda_{2^{s-1}}) & \text{if } 2 \leq w = s - 1 \text{ and } w = v_2(c) + 1, \\ (1) & \text{otherwise.} \end{cases}$$