5. Conservative transverse line fields

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.04.2024**

REMARK. The following result is also known (see the literature cited): if a convex closed curve intersects a curve, homothetic to J, at $2n$ points then it has at least $2n$ Minkowski vertices.

5. CONSERVATIVE TRANSVERSE LINE FIELDS

In this section we discuss the following problem: given a smooth strictly convex closed plane curve γ and a smooth transverse line field l along it, when does a parameterization $\gamma(t)$ exist such that the line $l(t)$ at point $\gamma(t)$ is generated by the acceleration vector $\gamma''(t)$ for all t ?

DEFINITION. A transverse line field along a closed plane curve, generated by the acceleration vectors for some parameterization of the curve, is called *conservative*.

Clearly, not every line field is conservative: consider, for example, a field of lines that everywhere make an acute angle with the curve. Theorem 0.1 provides a necessary condition: the envelope of the lines from a conservative line field has at least 4 cusps. Lemma 3.2 gives another one: there exist at least 2 tangent lines to this envelope through every point in the plane.

We start with the following situation. Let M^3 be a contact manifold and let $\widetilde{\gamma} \subset M$ be a closed smooth Legendrian curve. Recall that the characteristic line field η of a contact form λ is the field $Ker\ d\lambda$. Assume that the contact distribution along $\widetilde{\gamma}$ is coorientable; then it can be determined by a contact form. Let η be a line field along $\widetilde{\gamma}$, transverse to the contact distribution.

QUESTION. When does a contact form exist in a vicinity of $\widetilde{\gamma}$ for which η is the characteristic field?

When this is the case we call the field η *characteristic*.

Let λ be some contact form near $\widetilde{\gamma}$ and let v be a vector field along $\widetilde{\gamma}$ that generates the line field η. Consider the 1-form $(i_v\ d\lambda)/\lambda(v)$ and set

$$\beta(\widetilde{\gamma}, \eta) = \int_{\widetilde{\gamma}} \frac{i_v\ d\lambda}{\lambda(v)}\ .$$

THEOREM 5.1. *The number $\beta(\widetilde{\gamma}, \eta)$ does not depend on the choice of the contact form λ nor the vector field v. This number vanishes if and only if the field η is characteristic.*

Proof. Clearly, $(i_v\ d\lambda)/\lambda(v)$ does not change if v is multiplied by a nonvanishing function. Let $\lambda_1 = f\lambda$ with $f \neq 0$ be another contact form. Then $d\lambda_1 = df \wedge \lambda + f d\lambda$. One has

$$\int_{\tilde{\gamma}} \frac{i_v\ d\lambda_1}{\lambda_1(v)} = \int_{\tilde{\gamma}} \frac{f\ i_v\ d\lambda + df(v)\ \lambda - \lambda(v)\ df}{f\ \lambda(v)}$$
$$= \int_{\tilde{\gamma}} \frac{i_v\ d\lambda}{\lambda(v)} + \int_{\tilde{\gamma}} \frac{df(v)}{f\ \lambda(v)}\lambda - \int_{\tilde{\gamma}} \frac{df}{f}.$$

The second integral on the right hand side vanishes because $\tilde{\gamma}$ is a Legendrian curve, tangent to the kernel of $df(v)\lambda/f\lambda(v)$, and so does the third because df/f is an exact 1-form. Thus $\beta(\tilde{\gamma}, \eta)$ does not depend on the choices involved.

If η is characteristic for a contact form λ then $i_v\ d\lambda = 0$, so $\beta(\tilde{\gamma}, \eta) = 0$. Conversely, let $\beta(\tilde{\gamma}, \eta) = 0$. A neighbourhood of $\tilde{\gamma}$ in M is contactomorphic to a neighbourhood of the zero section in the space of 1-jets $J^1 S^1$ (see [A 3]). That is, there exist coordinates (x, y, z), $x \in S^1$, $y, z \in \mathbf{R}^1$ in which the contact structure is given by the 1-form $\lambda_0 = dz - y dx$, and $\tilde{\gamma}$ is the curve $y = z = 0$. Since η is transverse to the contact structure one may assume it to be generated by the vector field

$$v = a(x)\ \partial/\partial x + b(x)\ \partial/\partial y + \partial/\partial z,$$

where $a(x)$ and $b(x)$ are functions on the circle.

Then

$$\beta(\tilde{\gamma}, \eta) = \int_{\tilde{\gamma}} \frac{i_v\ d\lambda_0}{\lambda_0(v)} = -\int b(x)\ dx.$$

If $\beta(\tilde{\gamma}, \eta)$ vanishes then there exists a function $g(x)$ such that $b(x) = g'(x)$. Next, a direct computation shows that the characteristic line field of the contact form $e^{f(x,y,z)}\ \lambda_0$ is generated by the vector field

$$f_y\ \partial/\partial x\ - (f_x + y f_z)\ \partial/\partial y + (1 + y f_y)\ \partial/\partial z,$$

which equals, along $\tilde{\gamma}$,

$$u = f_y\ \partial/\partial x - f_x\ \partial/\partial y + \partial/\partial z.$$

Therefore, setting $f(x, y, z) = a(x)y - g(x)$, one has: $v = u$, and the field η is characteristic.

Thus the characteristic line fields constitute a codimension 1 subspace in the (infinite dimensional) space of line fields along $\tilde{\gamma}$, transverse to the contact structure.

Return to the situation at the beginning of the section. Let γ be a smooth strictly convex closed curve, cooriented inwards, and let l be a smooth transverse line field along γ. As before, $\widetilde{\gamma}$ is the Legendrian curve in the space of cooriented contact elements $ST^*\mathbf{R}^2$, corresponding to γ. For every point $x \in \gamma$ consider the family of cooriented contact elements along the line $l(x)$, parallel to the contact element of γ at x. This gives a line field η along $\widetilde{\gamma}$, a lift of the field l. The field η is transverse to the contact structure.

Choose a parameterization $\gamma(t)$, $0 \le t \le T$, and a vector field $u(t)$ along γ that generates the line field $l(t)$.

LEMMA 5.2. *One has:*

$$\beta(\widetilde{\gamma}, \eta) = \int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]}\, dt .$$

Proof. Let v be the lift of u to $ST^*\mathbf{R}^2$ that generates the field η. In Theorem 2.1 a Hamiltonian function H in $ST^*\mathbf{R}^2$ is constructed, associated with the parameterization $\gamma(t)$ (one does not need the assumption $[\gamma''(t), \gamma'''(t)] \ne 0$ here). The space $ST^*\mathbf{R}^2$ is identified with $\mathbf{R}^2 \times S$, where the star-shaped curve $S \subset (\mathbf{R}^2)^*$, the level curve of H, consists of the covectors $[\gamma'(t), \ \ ]$. The corresponding contact form λ is the restriction of the Liouville form pdq to $\mathbf{R}^2 \times S$. The curve $\widetilde{\gamma}$ is given by the formula:

$$\widetilde{\gamma}(t) = (\gamma(t), \ [\gamma'(t), \ \ ]) .$$

It follows that $\lambda(v(t)) = [\gamma'(t), u(t)]$. Likewise,

$$(i_{v(t)} d\lambda)(\widetilde{\gamma}'(t)) = (i_{v(t)} dp \wedge dq)(\widetilde{\gamma}'(t)) = [\gamma''(t), u(t)] .$$

Therefore

$$\int_{\widetilde{\gamma}} \frac{i_v \, d\lambda}{\lambda(v)} = \int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]}\, dt .$$

The lemma is proved.

In particular, the value of the integral

$$\int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]}\, dt$$

does not depend on the parameterization $\gamma(t)$ nor on the choice of the vector field $u(t)$. Denote this integral by $\alpha(\gamma, l)$.

LEMMA 5.3. *The line field l along γ is conservative if and only if the line field η along $\widetilde{\gamma}$ is characteristic.*

Proof. If l is generated by the vectors $\gamma''(t)$ then η consists of the characteristic directions of the contact form in $ST^*\mathbf{R}^2$, associated with the parameterization $\gamma(t)$ in Theorem 2.1 (cf. the proof of the preceding lemma).

Conversely, a contact form λ along $\widetilde{\gamma}$, whose characteristics are the lines η, is a field of covectors p along γ which vanish on the tangent lines to γ at the respective points. Define the parameterization $\gamma(t)$ by the condition: $[\gamma'(t), \quad] = p\ (\gamma(t))$ for all t. Then the contact form in $ST^*\mathbf{R}^2$, associated with this parameterization according to Theorem 2.1, coincides with λ along $\widetilde{\gamma}$. Therefore the lines $l(t)$ are generated by the vectors $\gamma''(t)$.

Combining Theorem 5.1, Lemma 5.2 and 5.3, one arrives at the following result (discovered in [T 2] and proved therein by a direct computation).

THEOREM 5.4. *A transverse line field l along a smooth strictly convex closed plane curve γ is conservative if and only if $\alpha(\gamma, l) = 0$.*

Thus conservative line fields constitute a codimension one subspace in the space of transverse line fields along a closed curve.

EXAMPLE. L. Guieu and V. Ovsienko studied the following situation in [G-O]. Given a smooth convex closed plane curve consider the field of lines connecting each point of the curve with a focus of its osculating conic at this point (see Example 2 in Section 3). This line field is conservative, and its envelope, called the gravitational caustic in [G-O], has at least 6 cusps.

Consider a curve γ with a transverse line field l. A (partial) diffeomorphism of the plane F takes γ to a new curve $F(\gamma)$ with the transverse line field $dF(l)$. The field $dF(l)$ does not have to be conservative even if l is.

EXAMPLE. Let γ be the unit circle, l consists of its normals, and F is given near γ in polar coordinates by the formula: $(\alpha, r) \to (\alpha + r, r)$. Then $F(\gamma) = \gamma$, and the lines $dF(l)$ make a constant acute angle with the circle.

However the following result holds (to answer a question by V. Arnold).

THEOREM 5.5. *Every projective transformation of the plane takes the conservative line fields to the conservative ones.*

Proof. Consider $\mathbf{R}^2$ as the plane $\{z = 1\}$ in Euclidean 3-space, and let

$$\pi : (x, y, z) \to (x/z, y/z)$$

be the projection of the half-space $\mathbf{R}^3_+ = \{z > 0\}$ on $\mathbf{R}^2$. Consider a parametrized curve $\Gamma(t) \subset \mathbf{R}^3_+$, and let $\gamma(t) = \pi\big(\Gamma(t)\big)$.

Claim: the field $(d\pi)\big(\Gamma''(t)\big)$ is conservative along the curve $\gamma(t)$.

Indeed, a direct computation (which is left to the reader) shows that

$$(d\pi)\big(\Gamma''(t)\big) = \gamma''(t) + 2\,\frac{z'(t)}{z(t)}\,\gamma'(t)\cdot$$

Therefore

$$\alpha\big(\gamma,\ (d\pi)(\Gamma''(t))\big) = -\int 2\,\frac{z'(t)}{z(t)}\,dt = -2\int d\ \log\ z(t) = 0\,.$$

The claim follows from Theorem 5.4.

Let A be a linear transformation of space. Then $F = \pi A : \mathbf{R}^2 \to \mathbf{R}^2$ is a projective transformation, and all projective transformations are obtained this way. Consider a curve $\gamma(t) \subset \mathbf{R}^2$, and let $l(t)$ be generated by the acceleration vectors $\gamma''(t)$. Let $\Gamma(t) = A\big(\gamma(t)\big)$; assume, without loss of generality, that $\Gamma(t) \subset \mathbf{R}^3_+$ One has: $\Gamma''(t) = A\big(\gamma''(t)\big)$, and it follows from the above claim that the field $(d\pi)\big(\Gamma''(t)\big)$ is conservative along the curve $\pi\big(\Gamma(t)\big)$. Thus the line field $dF(l)$ is conservative along the curve $F(\gamma)$.

REMARK. Theorem 5.5 shows that the notion of the conservative line fields along closed curves is a projective, and not an affine, one. Thus one hopes that the theory of this paper can be extended to spherical curves in the spirit of [A 5].

ACKNOWLEDGEMENTS. I am grateful to V. Arnold, S. Lvovsky, V. Ovsienko, G. Thorbergsson and M. Umehara for numerous stimulating discussions, and to G. Thorbergsson and M. Umehara for helping me to find my way in the immense literature on vertices (most of it, in German).

It is a pleasure to acknowledge the hospitality of the Max-Planck-Institut für Mathematik in Bonn. The research was supported in part by NSF grant DMS-9402732.

ADDED IN PROOF. A higher dimensional analog of conservative transverse line fields is studied in the author's paper "Exact transverse line fields and projective billiards in a ball", to appear in "Geometric and Functional Analysis".

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