Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	43 (1997)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	QUATERNARY CUBIC FORMS AND PROJECTIVE ALGEBRAIC THREEFOLDS
Autor:	SCHMITT, Alexander
Kapitel:	Introduction
DOI:	https://doi.org/10.5169/seals-63278

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

## Download PDF: 10.07.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

# QUATERNARY CUBIC FORMS AND PROJECTIVE ALGEBRAIC THREEFOLDS

by Alexander SCHMITT

# INTRODUCTION

As algebraic geometers, we are interested in a special kind of complex manifolds, namely in complex submanifolds of projective spaces. A submanifold X of  $\mathbf{P}_n$  is given as the common zero locus of a set of homogeneous polynomials such that the Jacobi matrix of these polynomials has rank  $n - \dim X$ at every point of X. We call such a manifold a *projective algebraic manifold*. The main goal is the classification of projective algebraic manifolds up to biholomorphic equivalence. Now, a projective algebraic manifold is in particular an oriented and closed topological manifold. Moreover, biholomorphic maps are orientation preserving homeomorphisms.

Thus, we obtain a natural approach to the classification of projective algebraic manifolds which can be stated for complex dimension 3 as follows:

Given a six-dimensional, closed, and oriented topological manifold X, describe all projective algebraic threefolds (up to biholomorphic equivalence) whose underlying topological manifold is orientation preservingly homeomorphic to X.

Of course, one does not have a general classification of the respective topological manifolds. However, if we restrict our attention to simply connected, six-dimensional, closed, and oriented topological manifolds with torsion free homology, there is a classification result in the sense of algebraic topology, due to C.T.C. Wall [Wa] and P.E. Jupp [Ju]. This means the classification of simply connected, six-dimensional, closed, and oriented topological manifolds with torsion free homology up to orientation preserving homeomorphy can be reduced to the classification of certain algebraic data, so called admissible systems of invariants.

The explicit classification of these algebraic data can be carried out in the case the second Betti number  $b_2$  is 1 [OV]. But already for  $b_2 = 2$ , the picture is rather complicated and not yet complete [Sch3]. So, it seems to be a rather hopeless task to classify systems of invariants for  $b_2 > 2$ . Thus, we restrict ourselves to the consideration of the most important part of the system of invariants of the simply connected, six-dimensional, closed, and oriented topological manifold X, the cup form

$$\varphi_X \colon S^3 H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$[a \otimes b \otimes c] \longmapsto (a \cup b \cup c)[X].$$

Here, [X] is the fundamental class of X. We remark that the assumptions we make on the manifold X imply that the whole cohomology ring of X is determined by  $\varphi_X$  and the third Betti number  $b_3(X)$ .

We can also replace  $\mathbb{Z}$  by  $\mathbb{R}$  or  $\mathbb{C}$  to obtain a weaker invariant. By our hypothesis,  $H^2(X, \mathbb{Z})$  is a free  $\mathbb{Z}$ -module, and  $H^2(X, R) = H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} R$ ,  $R = \mathbb{R}, \mathbb{C}$ . If we fix a basis for  $H^2(X, R)$ , we can identify  $\varphi_X$  with a homogeneous cubic polynomial. On the module of all homogeneous cubic polynomials in *b* variables, there is an action of  $GL_b(R)$  by substitution of variables. Hence, we obtain a coarse picture of the classification of simply connected, six-dimensional, closed, and oriented topological manifolds with  $b_2 = b$  if we determine the normal forms for cubic polynomials over  $\mathbb{Z}$  in *b* variables w. r. t. the action of  $GL_b(\mathbb{Z})$  and if we describe the set of forms  $\varphi_X$ , *X* being a topological manifold.

For the latter part, we remark that there is a simple criterion to check whether a given cubic polynomial over  $\mathbb{Z}$  is of the form  $\varphi_X$  or not (see [Sch2], Cor. 1). For example, this criterion is fulfilled if all coefficients are divisible by 6. The determination of normal forms is again very difficult. However, if we work over the field of complex numbers instead, results are known for up to b = 4 variables. The results for  $b \leq 3$  variables are easily accessible. On the other hand, the results for b = 4 are scattered in the literature of over 100 years. Hence, we have written an extensive summary of the theory of complex quaternary cubic forms. Being interested in (Cubic forms over  $\mathbb{Z}$ )/GL<sub>b</sub>( $\mathbb{Z}$ ), it is more reasonable to consider the action of  $\widetilde{SL}_b(\mathbb{C}) := \{m \in GL_b(\mathbb{C}) \mid \det(m) = \pm 1\}$ . To simplify things we will consider the action of SL<sub>b</sub>( $\mathbb{C}$ ) instead. This is the content of Part I.

In the second part, we treat the following weakened form of our original problem:

Which quaternary cubic forms can occur as cup forms of simply connected projective threefolds?

For the case  $b \le 3$ , we refer the reader to [OV]. In this part, we have collected a number of examples. We also show that there is a simply connected projective threefold with  $b_2 = 3$  whose cup form defines a plane cubic with a node, a problem which remained unsolved in [OV]. We conclude our notes by a brief summary of the author's results concerning the non-realizability of certain *real* cubic polynomials as cup forms of projective threefolds.

ACKNOWLEDGEMENTS. The results of this paper are part of the author's thesis [Sch1]. This thesis was written under the guidance of Prof. Ch. Okonek whom I wish to thank for many helpful discussions during the preparation of the thesis and this paper. The author wants to acknowledge financial support by AGE — Algebraic Geometry in Europe — Contract Number ERB CHRXCT 940557 (BBW 93.0187).

## I. QUATERNARY CUBIC FORMS

In this section, we will be concerned with the space  $S^3(\mathbb{C}^{4^{\vee}})$  of quaternary cubic forms on which  $SL_4(\mathbb{C})$  acts by substitution of variables. In particular, we will treat the following problems:

- 1) Find "good" representatives for the orbits in  $S^3(\mathbf{C}^{4^{\vee}})$ ;
- 2) Describe the categorical quotient  $S^3(\mathbb{C}^{4^{\vee}})//\mathrm{SL}_4(\mathbb{C})$ .

(The categorical quotient is an affine algebraic variety whose set of points is in natural bijection with the closed orbits in  $S^3(\mathbb{C}^{4^{\vee}})$ . A good introduction to this kind of constructions can be found in [Ne].)