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SQUARE-FREE TOWER OF HANOI SEQUENCES

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ABSTRACT. The sequence of moves in the optimal solution of the Tower of Hanoi with an arbitrary number of discs has recently been shown to lead to an example of an infinite square-free string over a six-letter alphabet by recourse to the theory of iterated morphisms. We present a direct approach to this result, using only properties of the Tower of Hanoi itself, which also reveals an implicit infinite square-free string with just five letters.

0. SQUARE-FREE STRINGS AND THE OLIVE SEQUENCE

Suppose you dispose of a reservoir of *letters* (or *symbols*), i.e. an at most countably infinite set $\mathcal{A}$, called an *alphabet*, and you are asked to construct a *word* (or *string*) of infinite length, i.e. a sequence $a \in \mathcal{A}^{\mathbf{N}}$, which does not contain any non-trivial immediate repetition, or *square*, i.e. there are no $m \in \mathbf{N}_0$ and $l \in \mathbf{N}$ such that

$$\forall k \in \{m+1, \ldots, m+l\} : a_{k+l} = a_k .$$

(For a concise survey on *square-free* words and their use in mathematics see [4].)

Assume $\mathcal{A} = \mathbf{N}$ and try an apparently economic approach, namely choose a_k as the smallest positive integer such that $(a_1, \ldots, a_k)$ does not contain any square. Then you come up with the following sequence:

$$(1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,\ldots)$$

A connoisseur of the mathematical theory of the Tower of Hanoi (TH) puzzle (see [13] for a comprehensive survey) will immediately recognize the pattern