

## §3. Presentations for F

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

§3. PRESENTATIONS FOR  $F$ 

Two presentations for  $F$  will be given in this section.

Now two groups  $F_1$  and  $F_2$  will be defined by generators and relations. The generators  $A, B, X_0, X_1, X_2, \dots$  will be referred to as *formal symbols*, as opposed to the functions defined above. Given elements  $x, y$  in a group,  $[x, y] = xyx^{-1}y^{-1}$ .

$$F_1 = \langle A, B : [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$$

$$F_2 = \langle X_0, X_1, X_2, \dots : X_k^{-1}X_nX_k = X_{n+1} \quad \text{for } k < n \rangle$$

**THEOREM 3.1.** *There exists a group isomorphism from  $F_1$  to  $F_2$  which maps  $A$  to  $X_0$  and  $B$  to  $X_1$ .*

*Proof.* There is a group homomorphism from the free group generated by the formal symbols  $A$  and  $B$  to  $F_2$  such that  $A$  maps to  $X_0$  and  $B$  maps to  $X_1$ . This homomorphism is surjective because  $X_n = X_0^{-(n-1)}X_1X_0^{n-1}$  for  $n \geq 2$ . To see that the defining relations of  $F_1$  are in the kernel of this homomorphism, note that

$$X_1^{-1}X_2X_1 = X_0^{-1}X_2X_0 \quad \text{and} \quad X_1^{-1}X_3X_1 = X_0^{-1}X_3X_0,$$

hence

$$[X_0X_1^{-1}, X_2] = 1 \quad \text{and} \quad [X_0X_1^{-1}, X_3] = 1,$$

hence

$$[X_0X_1^{-1}, X_0^{-1}X_1X_0] = 1 \quad \text{and} \quad [X_0X_1^{-1}, X_0^{-2}X_1X_0^2] = 1.$$

Thus to complete the proof of Theorem 3.1 it suffices to prove that there exists a group homomorphism from  $F_2$  to  $F_1$  which maps  $X_0$  to  $A$  and  $X_1$  to  $B$ . To prove this it in turn suffices, after setting  $Y_0 = A$  and  $Y_n = A^{-(n-1)}BA^{n-1}$  for  $n \geq 1$ , to prove that

$$(3.2) \quad Y_k^{-1}Y_nY_k = Y_{n+1} \quad \text{for } k < n.$$

A closely related statement is that

$$(3.3) \quad [A^{-1}B, Y_m] = 1 \quad \text{for } m \geq 3.$$

Lines (3.2) and (3.3) will be proved in this paragraph. To see that line (3.3) is true for  $m = 3$  note that

$$\begin{aligned} [AB^{-1}, A^{-1}BA] = 1 &\Rightarrow A^{-1}[AB^{-1}, A^{-1}BA]A = 1 \Rightarrow [B^{-1}A, A^{-2}BA^2] = 1 \\ &\Rightarrow [A^{-1}B, A^{-2}BA^2] = 1 \Rightarrow [A^{-1}B, Y_3] = 1. \end{aligned}$$

The same argument gives line (3.3) for  $m = 4$ . The following equations show that line (3.2) is true if line (3.3) is true for  $m = n - k + 2$ .

$$\begin{aligned} Y_n Y_k &= A^{-n+1} B A^{n-1} A^{-k+1} B A^{k-1} = A^{-k+2} A^{-(n-k+1)} B A^{n-k+1} A^{-1} B A^{k-1} \\ &= A^{-k+2} Y_{n-k+2} A^{-1} B A^{k-1} = A^{-k+2} A^{-1} B Y_{n-k+2} A^{k-1} \\ &= A^{-k+1} B A^{k-1} A^{-k+1} Y_{n-k+2} A^{k-1} = Y_k Y_{n+1} \end{aligned}$$

Thus line (3.2) is true for every positive integer  $n$  and  $k = n - 1$ . In particular,  $Y_3^{-1} Y_4 Y_3 = Y_5$ . Because line (3.3) is true for  $m = 3$  and  $m = 4$ , it follows that line (3.3) is true for  $m = 5$ . An obvious induction argument now gives line (3.3) for every  $m \geq 3$ . This proves lines (3.2) and (3.3).

The proof of Theorem 3.1 is now complete.  $\square$

**THEOREM 3.4.** *There exist group isomorphisms from  $F_1$  and  $F_2$  to  $F$  which map the formal symbols  $A, B, X_0, X_1, X_2, \dots$  to the corresponding functions in  $F$ .*

*Proof.* Example 1.2 shows that the interior of the support of the function  $AB^{-1}$  in  $F$  is disjoint from the supports of the functions  $A^{-1}BA$ ,  $A^{-2}BA^2$  in  $F$ , and so the functions  $A, B$  in  $F$  satisfy the defining relations of  $F_1$ . Thus there exists a group homomorphism from  $F_1$  to  $F$  which maps the formal symbols  $A, B$  to the corresponding functions in  $F$ . Corollary 2.6 shows that this group homomorphism is surjective. Theorem 3.1 shows that this surjective group homomorphism induces a surjective group homomorphism from  $F_2$  to  $F$  which maps the formal symbols  $X_0, X_1, X_2, \dots$  to the corresponding functions in  $F$ . To prove Theorem 3.4 it suffices to prove that this latter group homomorphism is injective.

It will be proved that this latter group homomorphism is injective in this paragraph. The defining relations of  $F_2$  imply that

$$X_k^{-1} X_n = X_{n+1} X_k^{-1}, \quad X_n^{-1} X_k = X_k X_{n+1}^{-1}, \quad X_n X_k = X_k X_{n+1} \quad \text{for } k < n.$$

It follows that every nontrivial element  $x$  of  $F_2$  can be expressed as a positive element times a negative element as in Corollary-Definition 2.7. If  $X_k$  occurs in both the positive and negative part of  $x$  but  $X_{k+1}$  occurs in neither, then because  $X_k X_{n+1} X_k^{-1} = X_n$  for  $n > k$ , it is possible to simplify  $x$  by deleting one occurrence of  $X_k$  from both the positive and negative part of  $x$  and replacing every occurrence of  $X_{n+1}$  in  $x$  by  $X_n$  for  $n > k$ . Thus every nontrivial element of  $F_2$  can be put in normal form as in Corollary-Definition 2.7. It follows from Corollary-Definition 2.7 that every nontrivial element of  $F_2$  maps to a nontrivial element of  $F$ , as desired.

This proves Theorem 3.4.  $\square$