

## §2. Tree diagrams

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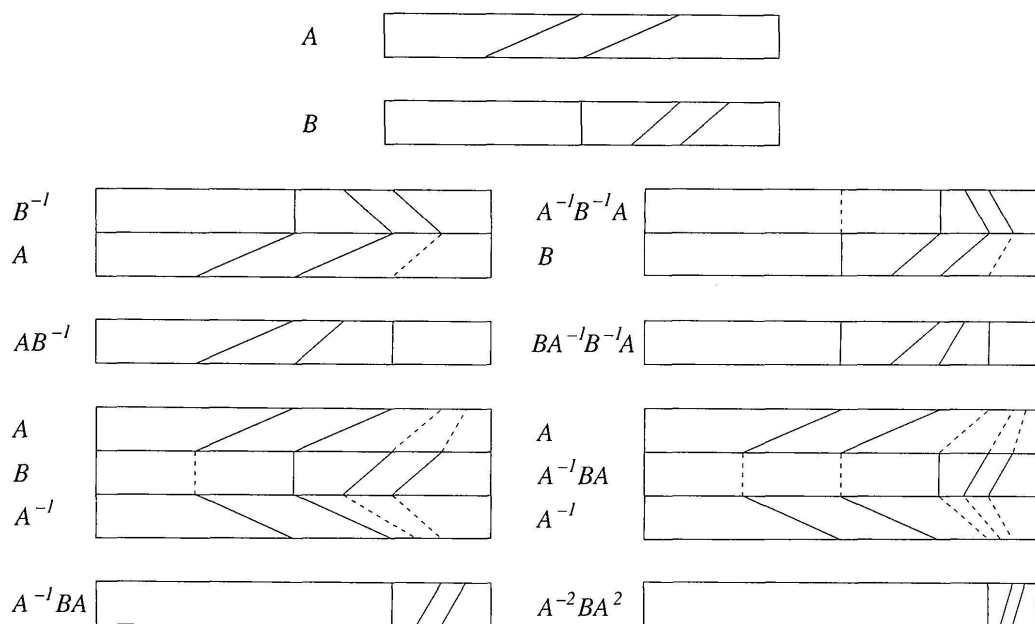


FIGURE 1  
The rectangle diagrams of some elements of  $F$



FIGURE 2  
The rectangle diagram of  $X_n$

### §2. TREE DIAGRAMS

The notion of tree diagram is developed in this section. Tree diagrams are useful for describing functions in  $F$ ; we first encountered them in [Bro1].

Define an *ordered rooted binary tree* to be a tree  $S$  such that i)  $S$  has a root  $v_0$ , ii) if  $S$  consists of more than  $v_0$ , then  $v_0$  has valence 2, and iii) if  $v$  is a vertex in  $S$  with valence greater than 1, then there are exactly two edges  $e_{v,L}$ ,  $e_{v,R}$  which contain  $v$  and are not contained in the geodesic from  $v_0$  to  $v$ . The edge  $e_{v,L}$  is called a *left edge* of  $S$ , and  $e_{v,R}$  is called a *right edge* of  $S$ . Vertices with valence 0 (in case of the trivial tree) or 1 in  $S$  will be called *leaves* of  $S$ . There is a canonical left-to-right linear ordering on the leaves of  $S$ . The *right side* of  $S$  is the maximal arc of right edges in  $S$  which begins at the root of  $S$ . The *left side* of  $S$  is defined analogously.

An *isomorphism* of ordered rooted binary trees is an isomorphism of rooted trees which takes left edges to left edges and right edges to right edges. An *ordered rooted binary subtree*  $S'$  of an ordered rooted binary tree  $S$  is an

ordered rooted binary tree which is a subtree of  $S$  whose left edges are left edges of  $S$ , whose right edges are right edges of  $S$ , but whose root need not be the root of  $S$ .

EXAMPLE 2.1. The right side of the ordered rooted binary tree in Figure 3 is highlighted. Its leaves are labeled  $0, \dots, 5$  in order.

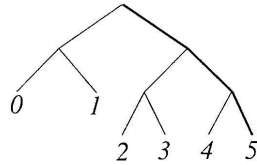


FIGURE 3

An ordered rooted binary tree with 6 leaves

Define a *standard dyadic interval* in  $[0, 1]$  to be an interval of the form  $[\frac{a}{2^n}, \frac{a+1}{2^n}]$ , where  $a, n$  are nonnegative integers with  $a \leq 2^n - 1$ .

There is a *tree of standard dyadic intervals*,  $\mathcal{T}$ , which is defined as follows. The vertices of  $\mathcal{T}$  are the standard dyadic intervals in  $[0, 1]$ . An edge of  $\mathcal{T}$  is a pair  $(I, J)$  of standard dyadic intervals  $I$  and  $J$  such that either  $I$  is the left half of  $J$ , in which case  $(I, J)$  is a left edge, or  $I$  is the right half of  $J$ , in which case  $(I, J)$  is a right edge. It is easy to see that  $\mathcal{T}$  is an ordered rooted binary tree. The tree of standard dyadic intervals is shown in Figure 4.

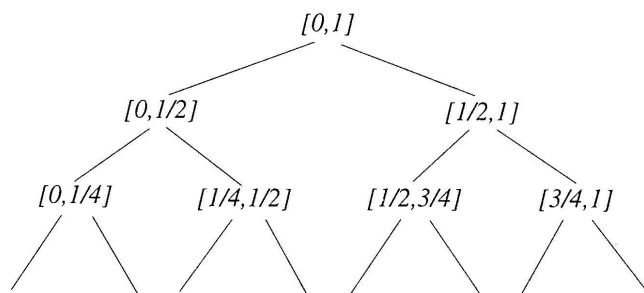


FIGURE 4

The tree  $\mathcal{T}$  of standard dyadic intervals

Define a  $\mathcal{T}$ -tree to be a finite ordered rooted binary subtree of  $\mathcal{T}$  with root  $[0,1]$ . Call the  $\mathcal{T}$ -tree with just one vertex the *trivial*  $\mathcal{T}$ -tree. For every nonnegative integer  $n$ , let  $\mathcal{T}_n$  be the  $\mathcal{T}$ -tree with  $n + 1$  leaves whose right side has length  $n$ .  $\mathcal{T}_3$  is shown in Figure 5.

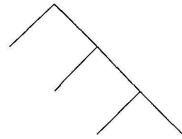


FIGURE 5  
The  $\mathcal{T}$ -tree  $\mathcal{T}_3$

Define a *caret* to be an ordered rooted binary subtree of  $\mathcal{T}$  with exactly two edges. Every caret has the form of the rooted tree in Figure 6.



FIGURE 6  
A caret

A partition  $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$  of  $[0, 1]$  determines intervals  $[x_{i-1}, x_i]$  for  $i = 1, \dots, n$  which are called the *intervals of the partition*. A partition of  $[0, 1]$  is called a *standard dyadic partition* if and only if the intervals of the partition are standard dyadic intervals.

It is easy to see that the leaves of a  $\mathcal{T}$ -tree are the intervals of a standard dyadic partition. Conversely, the intervals of a standard dyadic partition determine finitely many vertices of  $\mathcal{T}$ , and it is easy to see that these vertices are the leaves of their convex hull, which is a  $\mathcal{T}$ -tree. Thus there is a canonical bijection between standard dyadic partitions and  $\mathcal{T}$ -trees.

**LEMMA 2.2.** *Let  $f \in F$ . Then there exists a standard dyadic partition  $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$  such that  $f$  is linear on every interval of the partition and  $0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = 1$  is a standard dyadic partition.*

*Proof.* Choose a partition  $P$  of  $[0, 1]$  whose partition points are dyadic rational numbers such that  $f$  is linear on every interval of  $P$ . Let  $[a, b]$  be an interval of  $P$ . Suppose that the derivative of  $f$  on  $[a, b]$  is  $2^{-k}$ . Let  $m$  be an integer such that  $m \geq 0$ ,  $m + k \geq 0$ ,  $2^m a \in \mathbf{Z}$ ,  $2^m b \in \mathbf{Z}$ ,  $2^{m+k} f(a) \in \mathbf{Z}$ , and  $2^{m+k} f(b) \in \mathbf{Z}$ . Then  $a < a + \frac{1}{2^m} < a + \frac{2}{2^m} < a + \frac{3}{2^m} < \cdots < b$  partitions  $[a, b]$  into standard dyadic intervals, and  $f(a) < f(a) + \frac{1}{2^{m+k}} < f(a) + \frac{2}{2^{m+k}} < f(a) + \frac{3}{2^{m+k}} < \cdots < f(b)$  partitions  $[f(a), f(b)]$  into standard dyadic intervals. This easily proves Lemma 2.2.  $\square$

Formally, a *tree diagram* is an ordered pair  $(R, S)$  of  $\mathcal{T}$ -trees such that  $R$  and  $S$  have the same number of leaves. This is rendered diagrammatically as follows :

$$R \rightarrow S.$$

The tree  $R$  is called the *domain tree* of the diagram, and  $S$  is called the *range tree* of the diagram.

Suppose given  $f \in F$ . Lemma 2.2 shows that there exist standard dyadic partitions  $P$  and  $Q$  such that  $f$  is linear on the intervals of  $P$  and maps them to the intervals of  $Q$ . To  $f$  is associated the tree diagram  $(R, S)$ , where  $R$  is the  $\mathcal{T}$ -tree corresponding to  $P$  and  $S$  is the  $\mathcal{T}$ -tree corresponding to  $Q$ .

Because  $P$  and  $Q$  are not unique, there are many tree diagrams associated to  $f$ . Given one tree diagram  $(R, S)$  for  $f$ , another can be constructed by adjoining carets to  $R$  and  $S$  as follows. Let  $I$  be the  $n^{\text{th}}$  leaf of  $R$  for some positive integer  $n$ , and let  $J$  be the  $n^{\text{th}}$  leaf of  $S$ . Let  $I_1, I_2$  be the leaves in order of the caret  $C$  with root  $I$ , and let  $J_1, J_2$  be the leaves in order of the caret  $D$  with root  $J$ . Because  $f$  is linear on  $I$  and  $f(I) = J$ , it follows that  $f(I_1) = J_1$  and  $f(I_2) = J_2$ . Thus  $(R', S')$  is a tree diagram for  $f$ , where  $R' = R \cup C$  and  $S' = S \cup D$ .

In the other direction, if there exists a positive integer  $n$  such that the  $n^{\text{th}}$  and  $(n+1)^{\text{th}}$  leaves of  $R$ , respectively  $S$ , are the vertices of a caret  $C$ , respectively  $D$ , then deleting all of  $C$  and  $D$  but the roots from  $R$  and  $S$  leads to a new tree diagram for  $f$ . If there do not exist such carets  $C, D$  in  $R, S$ , then the tree diagram  $(R, S)$  is said to be *reduced*.

In this paragraph it will be shown that there is exactly one reduced tree diagram for  $f$ . Suppose that  $(R, S)$  is a reduced tree diagram for  $f$ . It is easy to see that if  $I$  is a standard dyadic interval which is either a leaf of  $R$  or not in  $R$ , then  $f(I)$  is a standard dyadic interval and  $f$  is linear on  $I$ . Conversely, if  $I$  is a standard dyadic interval such that  $f(I)$  is a standard dyadic interval and  $f$  is linear on  $I$ , then  $I$  is either a leaf of  $R$  or not in  $R$  because  $(R, S)$  is reduced. Thus  $R$  is the unique  $\mathcal{T}$ -tree such that a standard dyadic interval  $I$  is either a leaf of  $R$  or not in  $R$  if and only if  $f(I)$  is a standard dyadic interval and  $f$  is linear on  $I$ . This gives uniqueness of reduced tree diagrams.

Furthermore, if  $(R, S)$  is a tree diagram, then it is clear that there exists  $f \in F$  such that  $f$  is linear on every leaf of  $R$  and  $f$  maps the leaves of  $R$  to the leaves of  $S$ .

Thus there is a canonical bijection between  $F$  and the set of reduced tree diagrams.

EXAMPLE 2.3. Figure 7 shows the reduced tree diagrams for  $A$  and  $B$ .

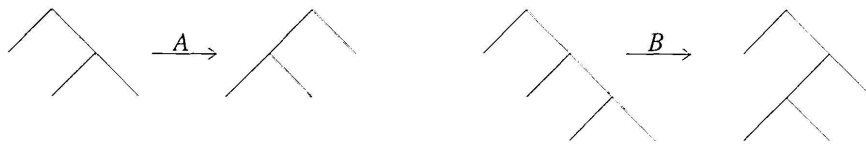


FIGURE 7

The reduced tree diagrams for  $A$  and  $B$

From Figure 2 it is not difficult to see that, for  $n \geq 0$ , the reduced tree diagram for  $X_n$  is the tree diagram in Figure 8.

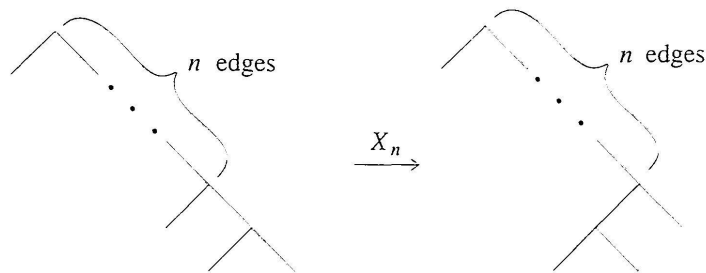


FIGURE 8

The reduced tree diagram for  $X_n$

It is easy to see that if  $(Q, R)$  is a tree diagram for a function  $f$  in  $F$  and  $(R, S)$  is a tree diagram for a function  $g$  in  $F$ , then  $(Q, S)$  is a tree diagram for  $gf$ .

The following definition prepares for Theorem 2.5, which makes the correspondence between functions in  $F$  and tree diagrams more precise. Define the *exponents* of a  $\mathcal{T}$ -tree  $S$  as follows. Let  $I_0, \dots, I_n$  be the leaves of  $S$  in order. For every integer  $k$  with  $0 \leq k \leq n$  let  $a_k$  be the length of the maximal arc of left edges in  $S$  which begins at  $I_k$  and which does not reach the right side of  $S$ . Then  $a_k$  is the  $k^{\text{th}}$  exponent of  $S$ .

EXAMPLE 2.4. Let  $S$  be the  $\mathcal{T}$ -tree shown in Figure 9.

The leaves of  $S$  are labeled  $0, \dots, 9$  in order, and the exponents of  $S$  in order are  $2, 1, 0, 0, 1, 2, 0, 0, 0, 0$ .

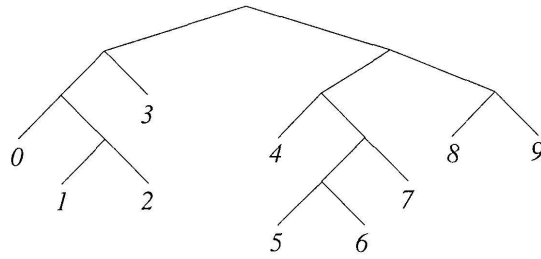


FIGURE 9  
The  $\mathcal{T}$ -tree  $S$

**THEOREM 2.5.** *Let  $R, S$  be  $\mathcal{T}$ -trees with  $n+1$  leaves for some nonnegative integer  $n$ . Let  $a_0, \dots, a_n$  be the exponents of  $R$ , and let  $b_0, \dots, b_n$  be the exponents of  $S$ . Then the function in  $F$  with tree diagram  $(R, S)$  is  $X_0^{b_0} X_1^{b_1} X_2^{b_2} \dots X_n^{b_n} X_n^{-a_n} \dots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$ . The tree diagram  $(R, S)$  is reduced if and only if i) if the last two leaves of  $R$  lie in a caret, then the last two leaves of  $S$  do not lie in a caret and ii) for every integer  $k$  with  $0 \leq k < n$ , if  $a_k > 0$  and  $b_k > 0$  then either  $a_{k+1} > 0$  or  $b_{k+1} > 0$ .*

*Proof.* To prove the first statement of the theorem, by composing functions it suffices to prove that the function with tree diagram  $(R, \mathcal{T}_n)$  is  $X_n^{-a_n} \dots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$ .

The proof of this will proceed by induction on  $a = \sum_{i=0}^n a_i$ . If  $a = 0$ , then  $R = \mathcal{T}_n$ , and the result is clear. Now suppose that  $a > 0$  and that the result is true for smaller values of  $a$ . Let  $m$  be the smallest index such that  $a_m > 0$ . Then there are ordered rooted binary subtrees  $R_1, R_2, R_3$  of  $R$  such that  $R$  has the form of the tree at the left of Figure 10.

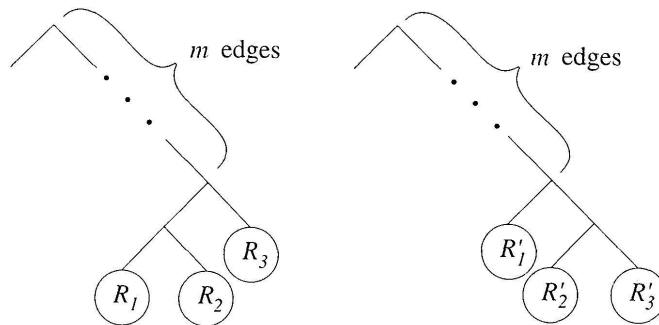


FIGURE 10  
The  $\mathcal{T}$ -trees  $R$  and  $R'$

Let  $R'$  be the  $\mathcal{T}$ -tree shown at the right of Figure 10, where  $R'_1, R'_2, R'_3$  are isomorphic with  $R_1, R_2, R_3$  as ordered rooted binary trees. According to Example 2.3, the function with tree diagram  $(R, R')$  is  $X_m^{-1}$ . If  $a'_0, \dots, a'_n$  are the exponents of  $R'$ , then  $a'_m = a_m - 1$  and  $a'_k = a_k$  if  $k \neq m$ . Thus

the induction hypothesis applies to  $R'$ , and so the function with tree diagram  $(R', \mathcal{T}_n)$  is  $X_n^{-a'_n} \cdots X_2^{-a'_2} X_1^{-a'_1} X_0^{-a'_0}$ . Again by composing functions, it follows that the function with tree diagram  $(R, \mathcal{T}_n)$  is  $X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$ , as desired.

The second statement of the theorem is now easy to prove.

This proves Theorem 2.5.  $\square$

COROLLARY 2.6. *Thompson's group  $F$  is generated by  $A$  and  $B$ .*

COROLLARY-DEFINITION 2.7. *Every nontrivial element of  $F$  can be expressed in unique normal form*

$$X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0},$$

where  $n, a_0, \dots, a_n, b_0, \dots, b_n$  are nonnegative integers such that i) exactly one of  $a_n$  and  $b_n$  is nonzero and ii) if  $a_k > 0$  and  $b_k > 0$  for some integer  $k$  with  $0 \leq k < n$ , then  $a_{k+1} > 0$  or  $b_{k+1} > 0$ . Furthermore, every such normal form function in  $F$  is nontrivial.

The functions in  $F$  of the form  $X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n}$  with  $b_k \geq 0$  for  $k = 0, \dots, n$  will be called *positive*. The positive elements of  $F$  are exactly those with tree diagrams having domain tree  $\mathcal{T}_n$  for some nonnegative integer  $n$ . Inverses of positive elements will be called *negative*.

LEMMA 2.8. *The set of positive elements of  $F$  is closed under multiplication.*

*Proof.* Let  $f$  and  $g$  be positive elements of  $F$ . Let  $(\mathcal{T}_m, R)$ , respectively  $(\mathcal{T}_n, S)$ , be tree diagrams for  $f$ , respectively  $g$ . If the right side of  $S$  has length  $k$ , then it is easy to see that  $fg$  has a tree diagram with domain tree  $\mathcal{T}_{n+\max\{m-k, 0\}}$ . Thus  $fg$  is positive. This proves Lemma 2.8.  $\square$

Fordham [Fo] gives a linear-time algorithm that takes as input the reduced tree diagram representing an element of Thompson's group  $F$  and gives as output the minimal length of a word in generators  $A$  and  $B$  representing that element. The algorithm can be modified to actually construct one, or all, minimal representatives. Fordham assigns a type to each caret of the tree pair; the minimal length is a simple function of the type sequences of the two trees.