

## 4. Centralisers of braid subgroups

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iterated  $2r$  times. Noting that  $I_p$  persists in  $w^*$  it is easy to argue that  $w(A * \sigma_j^{2r}) = w(A)$  is impossible; the contradiction.  $\square$

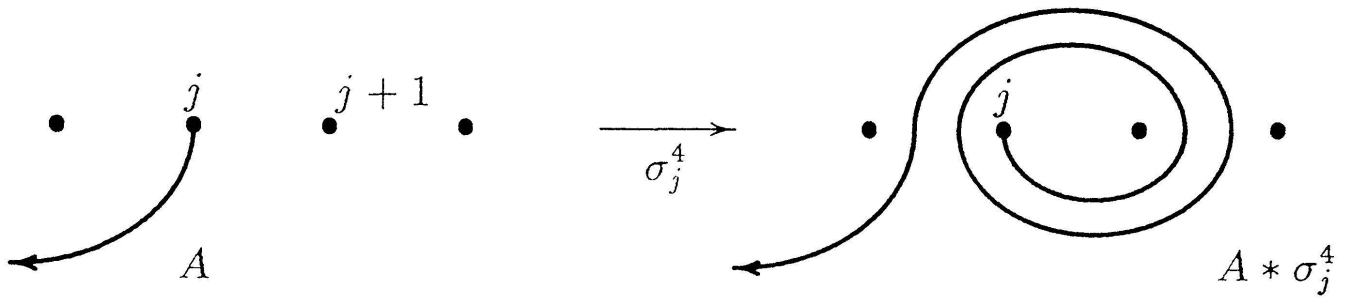


FIGURE 4

The action of  $*\sigma_j^{2r}$  on a  $(j, k)$ -arc in case  $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that  $(e) \Rightarrow (d) \Rightarrow (a)$ , and it is obvious that  $(a) \Rightarrow (c) \Rightarrow (b)$ . So it remains to establish that  $(b) \Rightarrow (e)$ . Thus we assume that, for some  $r \neq 0$ ,  $\sigma_j^r \beta = \beta \sigma_k^r$ . Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if  $\{j, j+1\} * \beta = \{k, k+1\}$ . Now, noting that  $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$  and that  $\sigma_k^r$  has a  $(k, k)$ -band, we conclude that there is a proper ribbon for  $\beta^{-1} \sigma_j^r \beta$  from  $[k, k+1] \times 0$  to  $[k, k+1] \times 1$ . Define  $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$ . Then we may assume (possibly after an isotopy) that the planes  $\mathbf{C} \times 1/3$  and  $\mathbf{C} \times 2/3$  cut the ribbon in the arcs  $A \times 1/3$  and  $A \times 2/3$ . Moreover, the middle third of the ribbon, and Proposition 1.1, imply that  $A * \sigma_j^r = A$ . By Lemma 3.2,  $A = [j, j+1]$  and the theorem is proved.  $\square$

#### 4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

**4.1 THEOREM.** *The centraliser in  $B_n$  of the generator  $\sigma_j$  is the subgroup of all braids which have  $(j, j)$ -bands. This subgroup is isomorphic to  $B_{n-1}^j \times \mathbf{Z}$  where  $B_{n-1}^j$  is the subgroup of  $B_{n-1}$  consisting of all  $(n-1)$ -braids whose permutations stabilise  $j$ .  $\square$*

The goal of this section is to describe the centraliser of  $B_r$  in  $B_n$ ,  $r \leq n$ , which we will call  $C(r, n)$ . Here  $B_r$  is the  $r$ -string braid group with its usual inclusion in  $B_n$ , namely as the subgroup generated by  $\sigma_1 \dots \sigma_{r-1}$ .

4.2 THEOREM. *The centraliser  $C(r, n)$  of  $B_r$  in  $B_n$  consists of all  $n$ -braids in which the first  $r$  strings lie on a ribbon, disjoint from the other strings, and which intersects  $\mathbf{C} \times 0$  and  $\mathbf{C} \times 1$  in exactly the straight line intervals from  $[1, r] \times 0$  and  $[1, r] \times 1$  (up to isotopy).*

*Proof.* A braid  $\beta$  is in  $C(r, n)$  if and only if it commutes with each  $\sigma_j$ ,  $1 \leq j \leq r - 1$ . Thus  $[j, j + 1] * \beta = [j, j + 1]$ ,  $1 \leq j \leq r - 1$  and so  $[1, r] * \beta = [1, r]$ , up to isotopy fixing  $\{1, \dots, n\}$ .  $\square$

It follows that  $C(r, n)$  consists of all  $n$ -braids constructible as follows. Let  $k = n - r + 1$  and consider the subgroup  $B_k^1$  of  $k$ -braids whose associated permutation fixes 1. Then replace the first string of a braid in  $B_k^1$  by  $r$  parallel strings lying on a ribbon along that string. The ribbon may be twisted by some integral multiple of  $2\pi$  (or  $\pi$  in the case  $r = 2$ ); such braids are precisely the central elements of  $B_r$ .

4.3 THEOREM. *The centraliser  $C(r, n)$  is isomorphic to the direct product  $B_{n-r+1}^1 \times \mathbf{Z}$ .*  $\square$

A PRESENTATION OF  $C(r, n)$ . In order to establish a set of generators and defining relations for  $C(r, n)$  we recall results of Chow [Ch] regarding  $B_k^1$ . This subgroup of  $B_k$  is generated by  $\sigma_2, \dots, \sigma_{k-1}$ , together with elements  $a_2, \dots, a_k$  defined by

$$a_i := \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-2}^{-1} \sigma_{i-1}^2 \sigma_{i-2} \cdots \sigma_2 \sigma_1.$$

These generators satisfy the usual braid relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

as well as the following, for  $i = 2, \dots, k - 1$ :

$$\begin{aligned} \sigma_i a_j \sigma_i^{-1} &= a_j, & j \neq i, i + 1 \\ \sigma_i a_i \sigma_i^{-1} &= a_{i+1} \\ \sigma_i a_{i+1} \sigma_i^{-1} &= a_{i+1}^{-1} a_i a_{i+1}. \end{aligned}$$

In fact these are *defining* relations for  $B_k^1$ . Chow also noted that the subgroup of  $B_k^1$  generated by the  $a_i$  is a normal subgroup (as is clear from the above relations), in fact a *free* group on the generators  $a_i$ , and that  $B_k^1$  could be regarded as the semidirect product of that free subgroup with the

subgroup generated by  $\sigma_2 \dots \sigma_{k-1}$ , the latter group clearly isomorphic with the braid group on  $k - 1$  strings.

Applying this to our situation, for each  $i = 1, \dots, n - r$ , let  $A_{r+i}$  be the  $n$ -braid resulting from replacing the first string of the  $k$ -braid  $a_i$ , defined above, by  $r$  parallel strings which lie on an untwisted band. Specifically,

$$A_{r+i} = (\sigma_r^{-1} \sigma_{r+1}^{-1} \cdots \sigma_{r+i-2}^{-1} \sigma_{r+i-1}) (\sigma_{r-1}^{-1} \sigma_r^{-1} \cdots \sigma_{r+i-3}^{-1} \sigma_{r+i-2}) \cdots (\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-1}^{-1} \sigma_i) \times (\sigma_i \sigma_{i-1} \cdots \sigma_1) (\sigma_{i+1} \sigma_i \cdots \sigma_2) \cdots (\sigma_{r+i-1} \sigma_{r+i-2} \cdots \sigma_r)$$

Also let  $C$  denote the well-known generator of the centre of the  $r$ -string braid group, namely  $C = \sigma_1$  if  $r = 2$  and in case  $r > 2$ :

$$C = (\sigma_1 \sigma_2 \cdots \sigma_{r-1})^r .$$

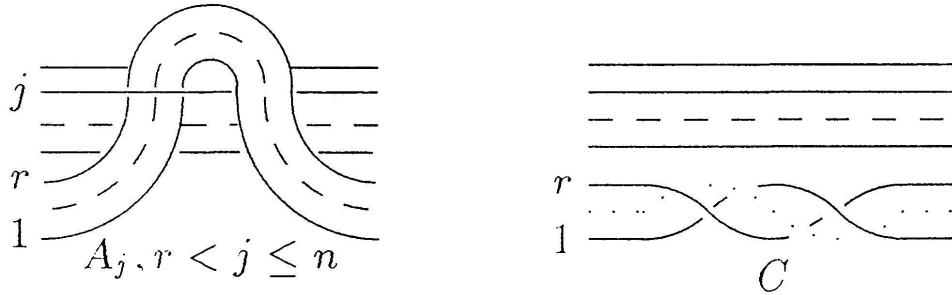


FIGURE 5

Special generators of  $C(r, n)$

4.4 THEOREM. *The centraliser  $C(r, n)$  of  $B_r$  in  $B_n$  has the generators:*

$$\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_{n-1}, A_{r+1}, \dots, A_n, C$$

and defining relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i A_j \sigma_i^{-1} &= A_j, & j \neq i, i + 1 \\ \sigma_i A_i \sigma_i^{-1} &= A_{i+1} \\ \sigma_i A_{i+1} \sigma_i^{-1} &= A_{i+1}^{-1} A_i A_{i+1} \\ C \sigma_i &= \sigma_i C \\ C A_i &= A_i C . \end{aligned}$$

(Subscripts ranging over all values for which the symbols are in the list of generators.)  $\square$