1. Introduction

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FACTOR EQUIVALENCE RESULTS FOR INTEGERS AND UNITS

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ABSTRACT. We give alternative proofs of two results of Fröhlich on the Galois module structure of the ring of integers and of the group of S-units in a Galois extension of number fields. We also point out applications to index computations in rings of integers and to class number relations.

1. INTRODUCTION

The purpose of this note is to give a brief presentation of basic factor equivalence results about the Galois module structure of the ring of integers and of the group of units in a Galois extension of number fields. Such results were first given by Nelson [12] and by Fröhlich [8, 9]. In [8] and [4, §3] these results are proved for abelian and for "admissible" Galois groups. It was shown later by Ritter and Weiss that all finite groups are "admissible" [14]. The proofs given below do not use any subtle representation-theoretic properties such as admissibility.

We set up the terminology in the next section. In Section 3 we show that the ring of integers in a Galois extension of number fields is "factor equivalent" to the group ring of the Galois group over the ring of integers of the base field. The proof uses the conductor discriminant formula, and it holds in the more general context of extensions of Dedekind domains of characteristic zero with separable residue field extensions.

In Section 4 the factor equivalence class of the lattice of units is expressed in terms of class numbers of intermediate fields. The proof uses zeta-functions and it holds for arbitrary Galois extensions of number fields.

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Finally, we give two applications in Section 5 that show how these results are related to more concrete questions in algebraic number theory. First we indicate how to do certain index computations for rings of integers in abelian extensions of number fields. For a bicyclic quartic field this implies that the lattice generated by its quadratic integers has index 2 in the ring of integers. Then we explain that the result for units gives a method to obtain class number inequalities between so-called "arithmetically equivalent" number fields.

2. Factorizability and factor equivalence

Let G be a finite group. A character of G is said to be rational if it is the character of a representation of G defined over $\mathbf{Q}$. Denote the additive group of rational characters of G by $R(G)$. One can view $R(G)$ as the Grothendieck group of finitely generated $\mathbf{Q}[G]$-modules. It is the free abelian group generated by the set $X(G)$ of isomorphism classes of irreducible $\mathbf{Q}[G]$-modules.

The trivial character 1_H on a subgroup H of G induces the permutation character $1_H^G \in R(G)$, corresponding to the G-module $\mathbf{Q}[G/H]$. Let $\mathcal{S}$ denote the set of subgroups of G and let T be an abelian group. We will use multiplicative notation for the group operation on T.

(2.1) Definition. *A function $f\colon \mathcal{S} \to T$ is said to be factorizable if for every collection of integers $(a_H)_{H\in\mathcal{S}}$ with $\sum_{H\in\mathcal{S}} a_H 1_H^G = 0$ we have $\prod_{H\in\mathcal{S}} f(H)^{a_H} = 1$.*

(2.2) Examples. If G is the Galois group of an extension of number fields L/K then Galois theory gives a bijection between $\mathcal{S}$ and the set of intermediate fields of L/K. For any parameter associated to number fields one thus obtains a function on $\mathcal{S}$, and one may wonder if it is factorizable. The discriminant, zeta-function, and the odd part of the number of roots of unity in a number field, are all factorizable. The p-part of the class number for $p \nmid [L:K]$ is also factorizable; cf. [18]. The fact that the parameter hR/w is factorizable is known as "Brauer's class number relations" (see Section 4). See Kani and Rosen [10, 11] for factorizability results for curves and Jacobians.

A function $f\colon \mathcal{S} \to T$ induces a group homomorphism $f_*\colon \mathbf{Z}[\mathcal{S}] \to T$, where $\mathbf{Z}[\mathcal{S}]$ is the free abelian group generated by $\mathcal{S}$. By definition f is factorizable if and only if f_* vanishes on the kernel of the homomorphism $r\colon \mathbf{Z}[\mathcal{S}] \to R(G)$ given by $H \mapsto 1_H^G$. For abelian groups G the map r is surjective. For every group G the image of r has finite index by Artin's