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## 4. THE LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS

Let $M$ be a compact $m$ dimensional manifold and $F$ a dimension $n$ foliation on $M$. Then $F$ is an $n$ dimensional subbundle of $TM$ such that for any two sections $X,\ Y \in C^\infty(F)$, $[X, Y] \in C^\infty(F)$. The Frobenius Theorem says that for each $x \in M$, there is a neighborhood $U$ of $x$ and a diffeomorphism

$$\phi : \mathbf{R}^n \times \mathbf{R}^q \to U \qquad n+q=m$$

so that for all $z \in \mathbf{R}^n \times \mathbf{R}^q$.

$$d\phi(T\mathbf{R}^n_z) = F_{\phi(z)}\,.$$

Such a $(U, \phi)$ is called a foliation chart. Given $x \in \mathbf{R}^q$, the submanifold $\phi\big(\mathbf{R}^n \times \{x\}\big)$ is called a plaque, and is denoted $P^U_x$. It is a local integral submanifold of $F$. The submanifold $\phi\big(\{0\} \times \mathbf{R}^q\big)$ is denoted $\mathbf{R}^q_U$ and is called the transverse submanifold of $(U, \phi)$.

A leaf $L$ of $F$ is a maximal integral (i.e. $TL_x = F_x$ for all $x \in L$) submanifold of $M$. Thus $\dim L = n$. The Frobenius Theorem implies that through each point $x$ in $M$, there passes a unique leaf, denoted $L_x$. Each leaf is a complete manifold of bounded geometry and the bounds are uniform for all leaves.

We now extend the Lefschetz Theorem for compact manifolds to a Lefschetz Theorem for foliations of a compact manifold. This is joint work with Connor Lazarov [HL 1]. In fact, we show how to improve the results of [HL 1] by removing the assumption that $F$ admits a transverse invariant metric. For a K-theory version of this result, see the thesis of M-T. Benameur, [Be].

Choose a smooth metric on $M$. This induces a smooth metric on each leaf $L$, and $L$ is complete with respect to this metric. Two different metrics on $M$ induce quasi-isometric metrics on $L$.

### HAEFLIGER FORMS

Let $\{U_i\}$ be a finite cover of $M$ by foliation charts. For $x \in U_i$, denote its plaque by $P^i_x$. If $U_i \cap U_j \neq \varnothing$ we define a local diffeomorphism $f_{ij}$ from $\mathbf{R}^q_{U_i}$ (hereafter denoted $\mathbf{R}^q_i$) to $\mathbf{R}^q_j$ as follows:

$$f_{ij}(x) = y \text{ if and only if } P^i_x \cap P^j_y \neq \varnothing\,.$$

The $f_{ij}$ generate the holonomy pseudogroup, denoted $H$, which acts on the transversal space $T = \cup_i \mathbf{R}^q_i$. We may (and do) assume that the $\mathbf{R}^q_i$ are disjoint.

Recall the following construction due to Haefliger [Ha]. Let $\Omega_c^k(T)$ be the space of bounded measurable complex valued $k$ forms on $T$ with compact support. Denote by $\Omega_c^k(T/H)$ the quotient of $\Omega_c^k(T)$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$ where $h \in H$ and $\alpha \in \Omega_c^k(T)$ has support contained in the range of $h$. Give $\Omega_c^k(T/H)$ the quotient topology of the usual sup norm topology on $\Omega_c^k(T)$. Note that $\Omega_c^k(T/H)$ does not depend of the choice of cover used to define it.

Denote by $\Omega^{p+k}(M)$ the space of bounded measurable complex valued $p+k$ forms on $M$. As the bundle $TF$ is oriented, there is a continuous open surjective linear map,

$$\int_F : \Omega^{p+k}(M) \to \Omega_c^k(T/H).$$

It is given as follows. Let $\omega \in \Omega^{p+k}(M)$ and let $\{\psi_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$. Set $\omega_i = \psi_i\omega$. We may integrate $\omega_i$ along the fibers of the submersion $\pi_i : U_i \to \mathbf{R}_i^q$ to obtain $\overline{\omega}_i \in \Omega_c^k(\mathbf{R}_i^q)$. Define $\int_F \omega$ to be the class of $\Sigma\overline{\omega}_i$ in $\Omega_c^k(T/H)$. It is independent of the choices made in defining it.

## Differential complexes on $M$ elliptic along $F$

A differential complex on $M$ along $F$ consists of:

a) a finite collection of finite dimensional complex vector bundles $E_0, \dots, E_k$ over $M$

b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \to C^\infty(E_{i+1})$$

with $d_{i+1} \cdot d_i = 0$

c) each $d_i$ differentiates only in leaf directions.

For the sake of simplicity we assume that each $d_i$ is first order.

Each of the classical complexes mentioned above (de Rham, Dolbeault, Signature and Twisted Spin) gives a leafwise complex on $M$ provided that the leaves have the required structures and that these structures are coherent from leaf to leaf (i.e. come from a global structure on $M$). For example, in the twisted Spin case, we require that the Spin structure on the leaves comes from a principal Spin($n$) bundle $P$ over $M$ with $P \times_{\mathrm{Spin}(n)} \mathbf{R}^n \simeq TF$, and that the leafwise auxiliary twisting bundle come from a bundle over $M$.

For a fixed leaf $L$, denote $E_i\big|_L$ by $E_i^L$ and by $C_0^\infty(E_i^L)$ the space of smooth sections of $E_i^L$ with compact support. The operator $d_i$ induces one, denoted also by $d_i$,

$$d_i : C_0^\infty(E_i^L) \to C_0^\infty(E_{i+1}^L)$$

and on $L$ we have the complex

$$0 \to C_0^\infty(E_0^L) \xrightarrow{d_0} C_0^\infty(E_1^L) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C_0^\infty(E_k^L) \to 0\,.$$

We say that the complex $(E,d)$ is elliptic along $F$ provided that for each leaf $L$, the above complex is elliptic. We assume that $(E,d)$ is elliptic along $F$.

### $L^2$ COHOMOLOGY OF $(E,d)$

Choose a smooth Hermitian metric on each bundle $E_i$ *over* $M$. These induce metrics on each $E_i^L$ and these metrics are unique up to quasi-isometry. Using these metrics we construct $d_i^* : C_0^\infty(E_{i+1}^L) \to C_0^\infty(E_i^L)$ just as we did before. We then construct

$$\Delta_i^L : C_0^\infty(E_i^L) \to C_0^\infty(E_i^L)$$

and we extend $\Delta_i$ to

$$\Delta_i^L : L^2(E_i^L) \to L^2(E_i^L)$$

just as before.

DEFINITION 4.1. *The $i$th $L^2$ cohomology of $(E,d)$ along the leaf $L$, denoted $H_L^i(E,d)$ is*

$$H_L^i(E,d) = \ker \Delta_i^L\,.$$

*The $i$th $L^2$ cohomology of $(E,d)$ is denoted $H^i(E,d)$ and it assigns to each leaf $L$ the $i$th cohomology of $(E,d)$ along $L, H_L^i(E,d)$.*

### SOME FACTS

1. $H_L^i(E,d)$ consists of smooth sections and $\dim_{\mathbf{C}} H_L^i(E,d)$ may be infinite but is always countable.
2. $\pi_L^i$, the projection of $L^2(E_i^L)$ onto $H_L^i(E,d)$, is a smoothing operator (on $L$) with smooth Schwartz kernel $k_L^i(x,y)$.
3. $k_L^i(x,y)$ is measurable as a function of $L$ and bounded independently of $L$. In particular, tr $k_L^i(x,x)$ is a bounded measurable function on $M$ whose restriction to each leaf $L$ is smooth.
4. Because of 3. above, we may define the dimension of $H^i(E,d)$ to be the zero dimensional Haefliger form

$$\dim\big(H^i(E,d)\big) = \int_F \operatorname{tr}\big(k_L^i(x,x)\big)\,dx\,,$$

where for any leaf $L$ we denote the volume form obtained from the metric on $L$ by $dx$. We may also define the Euler class of $(E,d)$ as

$$\chi(E,d) = \sum_{i=0}^{k}(-1)^i \dim H^i(E,d)\,.$$

GEOMETRIC ENDOMORPHISMS

Let $f : M \to M$ be a diffeomorphism and assume that for each leaf $L$ of $F$, $f(L) \subset L$. For each $i$, let

$$A_i : f^*E_i \to E_i$$

be a smooth bundle map. We assume that $T_i : C^\infty(E_i) \to C^\infty(E_i)$ where $(T_i s)(x) = A_{i,x} s\big(f(x)\big)$ satisfies

$$T_i d_{i-1} = d_{i-1} T_{i-1}\,.$$

The $T_i$ then induce maps

$$T_i^L : C_0^\infty(E_i^L) \to C_0^\infty(E_i^L)$$

satisfying

$$T_i^L d_{i-1} = d_{i-1} T_{i-1}^L\,.$$

We call such a family $T = (T_0, \dots, T_k)$ the geometric endomorphism of $(E,d)$ defined by $f$ and $A = (A_0, \dots, A_k)$. The $T_i^L$ extend to uniformly bounded linear maps

$$T_i^L : L^2(E_i^L) \to L^2(E_i^L)\,.$$

LEFSCHETZ NUMBER OF A GEOMETRIC ENDOMORPHISM

Set $T_{i,L}^* = \pi_i^L \cdot T_i^L \cdot \pi_i^L$ and denote its Schwartz kernel by $k_L^{T_i^*}(x,y)$. Then $k_L^{T_i^*}(x,y)$is globally bounded, smooth on $L \times L$, and measurable. Thus $\operatorname{tr}\big(k_L^{T_i^*}(x,x)\big)$ is a bounded measurable function on $M$ which is smooth on each leaf $L$. We define the Lefschetz class of the geometric endomorphism $T$ to be the Haefliger zero form

$$L(T) = \sum_{i=0}^{k}(-1)^i \int_F \operatorname{tr}\big(k_L^{T_i^*}(x,x)\big)\,dx\,.$$

For our Lefschetz Theorem we shall also need two restrictions on the fixed point set, $N$ of $f$. We require:

1. $N = \bigcup_\alpha N_\alpha$ is a finite disjoint union of closed, connected submanifolds $N_\alpha$, each transverse to $F$.
2. for each $x \in N \cap L = \bigcup_\alpha N_\alpha^L$ where $N_\alpha^L = N_\alpha \cap L$, $df_x$ has no eigen vector (in $TL_x$) with eigenvalue $+1$ in directions transverse (in $L$!) to $N_\alpha^L$.

Note in particular that $f = id_M$ satisfies these conditions.

### Fixed point indices

Let $\{U_i\}$ and $\{\psi_i\}$ be as above. Suppose that for each $L$ and $\alpha$ we are given a differential form $a_\alpha^L$ defined on $N_\alpha^L$. We define the Haefliger form $\int_N a$ as

$$\int_N a = \sum_i \sum_{N_\alpha^L \cap P_x^i \neq \phi} \int_{N_\alpha^L \cap P_x^i} \psi_i a_\alpha^L .$$

Note that for any plaque $P_x^i$, only a finite number of $N_\alpha^L$ satisfy $N_\alpha^L \cap P_x^i \neq \phi$. As $\int_{N_\alpha^L \cap P_x^i} \psi_i a_\alpha^L$ is a differential form on the transversal $\mathbf{R}_i^q$ of $U_i$, we may also consider it as a Haefliger form for $F$. As above, it is not difficult to show that the Haefliger form $\int_N a$ does not depend on the choices made in defining it.

THEOREM 4.2 (The Lefschetz Theorem for Foliations [HL 1]). *Let $M$, $F$, $f$, $T$, $A$ and $(E, d)$ be as above. To each $N_\alpha^L \subset N$ we may associate a differential form $a_\alpha^L$ which depends only on local data on $N_\alpha^L$ so that*

$$L(T) = \int_N a .$$

The proof follows the outline given above for the classical case, done leafwise. There are some very formidable technical obstacles, but these can be overcome (see [HL 1]).

If $(E, d)$ is the de Rham, Dolbeault, Signature or Twisted Spin complex of $F$, and $f = id_M$, and $T = id$, then $a_j^L$ is the usual local integrand formula (computed on each leaf, not on $M$) given by the Atiyah-Singer Index Theorem. We thus have an index theorem for foliated manifolds for these operators.

(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of $F$, his theorem is related to ours as the $L^2$ covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of $M$ which has one leaf (namely $M$), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e. $f \neq id_M$, $T = f^*$, $a_j^L$ is the usual local integrand (computed on the fixed point set in each leaf, not in $M$) given by the Atiyah-Singer $G$ Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer $G$ Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

## 5. Group Actions and the Lefschetz Theorem

Let $F$ be an oriented $2k$ dimensional foliation of a compact, oriented, Riemannian manifold $M$. Assume that $F$ admits a Spin$(2k)$ structure. That is, there is a principal Spin$(2k)$ bundle $P$ over $M$ and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF .$$

We may then construct the bundles $E^{\pm} = P \times_{\mathrm{Spin}(2k)} \Delta^{\pm}$. The leafwise Dirac operator $D^+$ is constructed using the Riemannian structure on the leaves of $F$ which is induced from $M$.

Let $G$ be a compact, connected Lie group acting by isometries on $M$, taking each leaf of $F$ to itself. $G$ then acts on $TF$. We assume that $G$ also acts on $P$ (commuting with the action of Spin$(2k)$) so that the induced action on $P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF$ is the given action on $TF$. $G$ then acts on the bundles $E^{\pm}$ and it commutes with the operator $D^+$, i.e. $G$ is a group of geometric endomorphisms of the complex $(E^{\pm}, D^+)$.

Recall the $\widehat{A}$ genus defined in Section 1.

Definition 5.1. *The $\widehat{A}$ genus of $F$ is the Haefliger zero form*

$$\widehat{A}(F) = \int_F \widehat{A}_{k/2}(TF) .$$

*In particular, if $k$ is odd, $\widehat{A}(F) = 0$.*

Note that we have defined $\widehat{A}(F)$ as the *zero th* order part of $\int_F \widehat{A}(TF)$. For an interpretation of the higher order terms of $\int_F \widehat{A}(TF)$, see [He].