**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 42 (1996)

**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CHARACTERISTIC CLASSES, ELLIPTIC OPERATORS AND COMPACT GROUP ACTIONS

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**DOI:** https://doi.org/10.5169/seals-87879

*L'Enseignement Mathématique, t. 42 (1996), p. 265–293*

# CHARACTERISTIC CLASSES, ELLIPTIC OPERATORS AND COMPACT GROUP ACTIONS

by James L. HEITSCH [1])

## 0. INTRODUCTION

Our aim in this paper is to introduce characteristic classes of vector bundles, to relate them to the indices of elliptic operators and the Lefschetz theorems for such operators, and finally to show how to use these to prove the non-existence of non-trivial actions preserving a Spin structure for compact connected Lie groups. Specifically, we will show how to prove the following.

THEOREM 5.2 ([HL 2]). *Let $F$ be an oriented foliation of a compact oriented manifold $M$ and assume that $F$ admits a Spin structure. If a compact connected Lie group acts non-trivially on $M$ as a group of isometries taking each leaf of $F$ to itself and preserving the Spin structure on $F$, then the $\widehat{A}$ genus of $F$ is zero.*

In [HL 1] and [HL 2], we assumed that $F$ admitted a transverse invariant measure. In this paper, we show how to remove that rather restrictive hypothesis by employing the Haefliger forms of $F$. In particular, the traces we use here have values in those forms rather than in the complex numbers. A transverse invariant measure defines a map from the Haefliger zero forms to the reals, and applying it to the traces we use in this paper produces the traces used in [HL 1] and [HL 2]. Note in particular that all the results of [HL 1] and [HL 2] are still valid even if $F$ does not admit a transverse invariant measure. One need merely ignore the transverse invariant measure and interpret the traces used as taking values in the Haefliger zero forms instead of the complex

[1]) Partially supported by NSF Grant DMS 9400676.

numbers. For another application of Haefliger forms and their cohomology, see [He].

An immediate corollary is the following theorem of Atiyah and Hirzebruch.

THEOREM 5.3 ([AH]). *Let $M$ be a compact connected, oriented manifold which admits a Spin structure. If a compact connected Lie group acts non-trivially on $M$, then the $\widehat{A}$ genus of $M$ is zero.*

Theorem 5.2 is an application of the Lefschetz fixed point theorem for complexes elliptic along the leaves of a foliated manifold. We explain the classical Lefschetz theorem for elliptic complexes and give an outline of how to prove it. The original proofs of this theorem relied on the fact that the underlying manifold was compact. We outline a proof which does not rely on that fact, and so can be generalized to complexes defined along the leaves of a compact foliated manifold. Note that such leaves are in general not compact, but the fact that they come from a foliation of a compact manifold means that they have uniformly bounded geometry. It is this property which allows us to prove the foliation version of the Lefschetz theorem. We then show how the Lefschetz theorem leads to Theorem 5.2. Finally, we give a brief explanation of a very general rigidity theorem conjectured by Witten and proven by Bott and Taubes.

This paper is based on lectures given at the conference Actions Différentiables de Groupes Compacts, Espaces d'Orbites et Classes Caractéristiques, held at the Université des Sciences et Techniques du Languedoc in Montpellier in January, 1994. The author wishes to thank the organizers, especially Daniel Lehmann and Pierre Molino, for extending the invitation to him to speak at the conference and for making his stay in Montpellier so pleasant.

## 1. CHARACTERISTIC CLASSES AND MULTIPLICATIVE SEQUENCES

All objects considered in this paper will be smooth. Let $E$ be an $n$ dimensional complex vector bundle over the real manifold $M$. Denote the space of smooth sections of $E$ by $C^\infty(E)$. A connection on $E$ is a linear map $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ satisfying

$$\nabla(f \cdot \sigma) = df \otimes \sigma + f \cdot \nabla\sigma$$

for any $\sigma \in C^\infty(E)$ and $f \in C^\infty(M)$, the smooth functions on $M$. $T^*M$ denotes the cotangent bundle of $M$.

If $\sigma_1, \dots, \sigma_n$ is a local basis of $C^\infty(E)$ on an open set $U$, the local connection form $\theta_U$ (which is an $n \times n$ matrix of 1 forms) is defined by

$$\nabla \sigma_i = \sum_{j=1}^{n} \theta^i_{U,j} \otimes \sigma_j .$$

The local curvature form $\Omega_U$ is the $n \times n$ matrix of 2 forms

$$\Omega_U = d\theta_U - \theta_U \wedge \theta_U .$$

It is not difficult to show that if $\tau_1, \dots, \tau_n$ is another local basis on the open set $V$ with $\tau_i = \sum_{j=1}^{n} g_{ij}\sigma_j$ on $U \cap V$, with $g_{ij} \in C^\infty(U \cap V)$, then on $U \cap V$

$$\Omega_U = g\Omega_V g^{-1} \tag{1.1}$$

where $g = [g_{ij}]$.

Now consider the local differential form on $U$, $\det\left(I - \frac{1}{2\pi i}\Omega_U\right)$. Because of (1.1), this is actually a well defined *global* form on $M$. This form depends only on $\nabla$ and it is closed, so it defines a cohomology class $c(E)$, the total Chern class of $E$, which actually takes values in the real de Rham cohomology of $M$. This class depends only on $E$ and we may write

$$c(E) = 1 + c_1(E) + \cdots + c_n(E)$$

where $c_k(E) \in H^{2k}(M, \mathbf{R})$ is the $k$th Chern class of $E$.

If $E$ is an $n$ dimensional real vector bundle over $M$, it is easy to show that $c_{2k+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$, and the $k$th Pontrjagin class of $E$ is defined to be

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbf{R}} \mathbf{C}) .$$

For more on this see [KN] and [M].

Let $Q(z) = \sum_{i=0}^{\infty} b_i z^i$ be a formal power series in $z$. Associated to $Q$ is the *multiplicative sequence* $K = (K_0, K_1, K_2, \dots)$ where each $K_j$ is a polynomial in $j$ indeterminants, $K_j(\sigma_1, \dots, \sigma_j)$ given as follows. Denote by $Q_j$ the degree $j$ part of $Q(z_1) \dots Q(z_j)$, where each $z_i$ has degree 1. $Q_j$ is a symmetric polynomial in the $z_i$ so it can be written as a polynomial in the elementary symmetric polynomials $\sigma_1, \dots, \sigma_j$ in $z_1, \dots, z_j$, i.e.

$$Q_j = K_j(\sigma_1, \dots, \sigma_j) .$$

For example, if $Q(z) = 1 + z$, then $Q_j = z_1 \dots z_j = \sigma_j$ and $K_j(\sigma_1, \dots, \sigma_j) = \sigma_j$.

If $Q(z)$ is an even power series, $Q(z) = \sum_{i=0}^{\infty} b_{2i} z^{2i}$, then the degree $2j$ part of $Q(z_1) \dots Q(z_j)$ can be written as a polynomial in the elementary symmetric polynomials $\gamma_1, \dots, \gamma_j$ in $z_1^2, \dots, z_j^2$. We set $K_j(\gamma_1, \dots, \gamma_j)$ to be this polynomial.

DEFINITION 1.2. *(a) Let $E$ be an $n$ dimensional complex vector bundle over $M$ and $Q(z)$ a formal power series with associated multiplicative sequence $K = (K_0, K_1, \dots)$. The $K$ genus of $E$, $K(E)$ is the de Rham cohomology class*

$$K(E) = \sum_{j=0}^{\infty} K_j\big(c_1(E), \dots, c_j(E)\big).$$

*(as $K_j\big(c_1(E), \dots, c_j(E)\big) \in H^{2j}(M, \mathbf{R})$, this is actually a finite sum).*

*(b) Let $E$ be an $n$ dimensional real vector bundle over $M$ and $Q(z)$ an even formal power series with associated multiplicative sequence $K = (K_0, K_1, \dots)$. Then the $K$ genus of $E$ is $K(E) = \sum\limits_{j=0}^{\infty} K_j\big(p_1(E), \dots, p_j(E)\big)$.*

$K$ is called a multiplicative sequence because $K(E_1 \oplus E_2) = K(E_1) \cdot K(E_2)$.

IMPORTANT EXAMPLES

1. $Q(z) = 1 + z$. Then

$$K(z) = 1 + c_1(E) + c_2(E) + \cdots = c(E).$$

2. $Q(z) = z/\tanh(z)$, which is even and gives the $L$ genus of Hirzebruch.

Recall that the signature $\operatorname{Sign}(M)$ of a compact oriented $4k$ dimensional manifold $M$ is the signature of the quadratic form on $H^{2k}(M, \mathbf{R})$ given by $\alpha, \beta \longmapsto \int_M \alpha \cdot \beta$.

THEOREM 1.3 (Hirzebruch [H]).

$$\operatorname{Sign}(M) = \int_M L_k\big(p_1(TM), \dots, p_k(TM)\big),$$

*where $TM$ is the tangent bundle of $M$.*

Thus $\operatorname{Sign}(M)$ is completely determined by the Pontrjagin classes of $M$.

3. $Q(z) = z/(1 - e^{-z})$, gives the Todd genus $Td$.

Let $M$ be a compact, complex $n$ dimensional manifold and $V$ a holomorphic vector bundle over $M$. Recall the Dolbeault complex of $V$

$$0 \to A^{0,0}(V) \xrightarrow{\bar\partial_0} A^{0,1}(V) \xrightarrow{\bar\partial_1} \cdots \xrightarrow{\bar\partial_{n-1}} A^{0,n}(V) \to 0$$

where $A^{0,q}(V)$ is the space of differential forms on $M$ of type $0,q$ with coefficients in $V$. $H^q(M,V)=$ kernel $\bar\partial_q/$ image $\bar\partial_{q-1}$ and it is finite dimensional (and isomorphic to $H^q\big(M,\mathcal{O}(V)\big)$ where $\mathcal{O}(V)$ is the sheaf of germs of holomorphic sections of $V$).

A fundamental invariant of $V$ is its Euler class

$$\chi(M,V)=\sum_{q=0}^{n}(-1)^q\dim H^q(M,V).$$

The Riemann-Roch problem is to calculate this integer from topological information about $M$ and $V$. The solution is given as follows. Suppose dimension $V=k$. $e^{z_1}+\cdots+e^{z_k}$ is symmetric in the $z_i$ so may be written as a power series in $\sigma_1,\dots,\sigma_k$, i.e. $\sum_{i=1}^{k}e^{z_i}=k+\mathrm{ch}_1(\sigma_1)+\mathrm{ch}_2(\sigma_1,\sigma_2)+\cdots$ where

$$\mathrm{ch}_j\big(\sigma_1(z_1,\dots,z_j),\dots,\sigma_j(z_1,\dots,z_j)\big)=\sum_{i=1}^{k}z_i^j/j!\,.$$

Set $\mathrm{ch}(V)=k+\mathrm{ch}_1\big(c_1(V)\big)+\mathrm{ch}_2\big(c_1(V),c_2(V)\big)+\cdots$ .

THEOREM 1.4 (The Riemann-Roch Theorem, [AS]).

$$\chi(M,V)=\int_M Td(TM)\cdot\mathrm{ch}(V).$$

Thus $\chi(M,V)$ is completely determined by the Chern classes of $M$ and $V$.

4. $Q(z)=(z/2)/\sinh(z/2)=z/(e^{z/2}-e^{-z/2})$ is an even function and gives the $\widehat{\mathcal{A}}$ genus.

Recall that Spin$(n)$ is the simply connected double cover of $SO(n)$. A Spin structure on an oriented Riemannian manifold $M$ of dimension $n$ is a principal Spin$(n)$ bundle $P$ over $M$ and an isomorphism of oriented bundles

$$P\times_{\mathrm{Spin(n)}}\mathbf{R^n}\simeq TM.$$

Spin$(n)$ has a complex representation space $\Delta$ of dimension $2^n$. See [ABS], [LM]. If $n=2k$, $\Delta$ may be written as $\Delta=\Delta^+\oplus\Delta^-$ where $\Delta^\pm$ are irreducible representations of dimension $2^{n-1}$. Set $E^\pm=P\times_{\mathrm{Spin}(n)}\Delta^\pm$. The metric connection on $M$ defines one on $E=E^+\oplus E^-$, denoted $\nabla$. The Dirac operator $D^+:C^\infty(E^+)\to C^\infty(E^-)$ is defined as follows. Let $c:C^\infty\big(T^*M\otimes E\big)\to C^\infty(E)$ be Clifford multiplication (we identify $T^*M$ with $TM$ using the metric on $M$). Then $D=c\cdot\nabla:C^\infty(E)\to C^\infty(E)$ and

$D$ maps $C^\infty(E^+)$ to $C^\infty(E^-)$ and vice-versa, since $c$ does. Thus we may write

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \tag{1.5}$$

where $D^\pm : C^\infty(E^\pm) \to C^\infty(E^\mp)$. (See [AS] or [LM]). Now $\ker D^+$ and $\operatorname{coker} D^+ (\simeq \ker D^-)$ are finite dimensional and the Spinor index of $M$,

$$\operatorname{Spin}(M) = \dim \ker D^+ - \dim \operatorname{coker} D^+ .$$

THEOREM 1.6 ([AS]). *If $M$ is a Spin manifold of* $\dim 2k$ *then*

$$\operatorname{Spin}(M) = \int_M \widehat{A}(TM) .$$

In particular, if $2k \equiv 2(4)$, then Spin(M) $= 0$ as $\widehat{A}(TM)$ involves only the Pontrjagin classes of $M$ and these occur only in dimensions $\equiv 0(4)$.

More generally, we may construct the twisted spinor complex. For this, let $F$ be a complex bundle over $M$ with hermitian metric and connection. Combining the connection on $E$ with that on $F$ we obtain a connection on $E \otimes F$. Composing this with Clifford multiplication

$$c : C^\infty(T^*M \otimes E \otimes F) \to C^\infty(E \otimes F)$$

we obtain the twisted Dirac operator $D_F$ on $E \otimes F$. As before $D_F$ interchanges $E^+ \otimes F$ and $E^- \otimes F$ and we get the *twisted Spin complex*

$$0 \to C^\infty(E^+ \otimes F) \xrightarrow{D_F^+} C^\infty(E^- \otimes F) \to 0 .$$

The kernel and cokernel of $D_F^+$ are finite dimensional and the twisted spinor index is

$$\operatorname{Spin}(M, F) = \dim \ker D_F^+ - \dim \operatorname{coker} D_F^+ .$$

THEOREM 1.7 ([AS]).

$$\operatorname{Spin}(M, F) = \int_M \widehat{A}(TM) \cdot \operatorname{ch}(F) .$$

## 2. The index of an elliptic complex

An *elliptic complex* $(E, d)$ over a closed, oriented, $n$ dimensional Riemannian manifold $M$ consists of:

a) a finite collection of finite dimensional complex vector bundles

$$E_0, E_1 \dots, E_k$$

b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \to C^\infty(E_{i+1})$$

c) The operators $d_i$ are required to satisfy

$$d_{i+1} \cdot d_i = 0$$

and an additional technical condition called ellipticity. This condition makes possible the construction a virtual bundle, i.e. the formal difference of two vector bundles, over $TM$ which carries a great deal of information about $(E, d)$. This virtual bundle $\sigma(E, d)$ is called the symbol of $(E, d)$ and it defines a class $[\sigma(E, d)]$, also called the symbol, in the K theory with compact supports of $TM$.

Examples

1. The de Rham complex, where

$$\begin{aligned} T^*_{\mathbf{C}}M &= \text{complexified cotangent bundle of } M \\ E_i &= \Lambda^i T^*_{\mathbf{C}}M \text{ the } ith \text{ exterior power of } T^*_{\mathbf{C}}M \\ C^\infty(E_i) &= \text{smooth complex } i \text{ forms on } M \\ d_i &= \text{the usual exterior derivative} \end{aligned}$$

2. The Dolbeault complex
3. The Signature complex (see [AS])
4. The twisted Spin complex.

SOME FACTS ABOUT ELLIPTIC COMPLEXES

Set $H^i(E,d) = \ker d_i / \text{image } d_{i-1}$. If $M$ is compact, then $\dim H^i(E,d) < \infty$, and we may define

$$\text{Index}(E,d) = \sum_{i=0}^{k} (-1)^i \dim H^i(E,d).$$

This is a very important invariant. Special cases of $(E,d)$ yield the

1. Euler class $\chi(M)$ of $M$ (de Rham complex)
2. Signature of $M$ (Signature complex)
3. Euler class $\chi(M,V)$ (Dolbeault complex)
4. $\widehat{\mathcal{A}}$ genus of $M$ (Spin complex).

The Atiyah-Singer Index Theorem tells how to compute this invariant from topological information about $M$ and $(E,d)$. In particular, it says

THEOREM 2.1 ([AS]).

$$\text{Index}(E,d) = \int_M Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \text{ch}\,\left(\sigma(E,d)\right).$$

The theorems quoted above are all special cases of this theorem. We now give an idea of how to prove this deep and important theorem.

On each $E_i$ choose an Hermitian inner product denoted $(\ ,\ )_i$. This induces an inner product $\langle\ ,\ \rangle_i$ on $C^\infty(E_i)$ by the formula

$$\langle s_1, s_2 \rangle_i = \int_M \left(s_1(x), s_2(x)\right)_i dx.$$

Using $\langle\ ,\ \rangle_i$ we define the adjoints

$$d_i^* : C^\infty(E_i) \to C^\infty(E_{i-1})$$

by

$$\langle s_1, d_i^* s_2 \rangle_{i-1} = \langle d_{i-1} s_1, s_2 \rangle_i$$

where

$$s_1 \in C^\infty(E_{i-1}), \quad s_2 \in C^\infty(E_i).$$

The Laplacian $\Delta_i : C^\infty(E_i) \to C^\infty(E_i)$ is defined by

$$\Delta_i = d_{i-1} d_i^* + d_{i+1}^* d_i,$$

and it extends to a densely defined operator of $L^2(E_i)$, the space of $L^2$ sections of $E_i$, as follows. $\Delta_i$ is a diagonalizable operator, and any eigenvalue $\lambda$ of $\Delta_i$ must be real and nonnegative. If $M$ is compact there is a sequence of real numbers

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots , \quad \lim_{j \to \infty} \lambda_j = \infty$$

such that for each $E_i$ there is a sequence of finite dimensional subspaces of $C^\infty(E_i)$, denoted

$$E_i(\lambda_0), E_i(\lambda_1), E_i(\lambda_2), \ldots$$

so that for any $s \in E_i(\lambda_j)$

$$\Delta_i s = \lambda_j s .$$

In addition

$$L^2(E_i) = \bigoplus_{j=0}^{\infty} E_i(\lambda_j) .$$

Thus each element in $L^2(E_i)$ can be written as a (possibly infinite) sum of eigen functions and we may think of $\Delta_i$ as the infinite diagonal matrix

$$\begin{bmatrix} 0 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0 & & & & & & & \\ & & & \lambda_1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \lambda_1 & & & & \\ & & & & & & \lambda_2 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \lambda_2 & \\ & & & & & & & & & \ddots \end{bmatrix} .$$

OTHER PROPERTIES OF $\Delta_i$

1) $E_i(\lambda_0) = \ker \Delta_i \subset \ker d_i$ and the inclusion of $E_i(\lambda_0)$ in $\ker d_i$ induces an isomorphism

$$E_i(\lambda_0) \simeq H^i(E, d) ,$$

so

$$\operatorname{Index}(E, d) = \sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_0) .$$

2) For each $\lambda_j > 0$, the sequence

$$0 \to E_0(\lambda_j) \xrightarrow{d_0} E_1(\lambda_j) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} E_k(\lambda_j) \to 0$$

is exact.

As a corollary, we have immediately

$$\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_j) = 0$$

for all $\lambda_j > 0$. These results rely on the fact that $M$ is compact. For a general reference for the above facts, see [Wa].

The fact that $\Delta_i$ is diagonal implies that for any function $f : \mathbf{R} \to \mathbf{R}$, we may define

$$f(\Delta_i) : L^2(E_i) \to L^2(E_i)$$

by : for each $s \in E_i(\lambda_j)$ set $f(\Delta_i)s = f(\lambda_j)s$. i.e. the "matrix" of $f(\Delta_i)$ is

$$\begin{bmatrix} f(0) & & & & & & \\ & \ddots & & & & & \\ & & f(0) & & & & \\ & & & f(\lambda_1) & & & \\ & & & & \ddots & & \\ & & & & & f(\lambda_1) & \\ & & & & & & \ddots \end{bmatrix}$$

Note also that if $f(x)$ goes to zero rapidly enough as $x \to \infty$, then the trace of $f(\Delta_i)$, thought of as the usual trace applied to the infinite matrix above, will be a finite number. In this case, we say $f(\Delta_i)$ is of trace class. See [RS].

We are interested in the family of functions $f_t(x) = e^{-tx}$, $t > 0$. In fact, even if $M$ is not compact, $e^{-t\Delta}$ makes sense and we have

THEOREM 2.2 (Seeley, [S]). *For $t > 0$, $e^{-t\Delta_i}$ is a smoothing operator on $L^2(E_i)$ and so if $M$ is compact it is of trace class.*

Let $\pi_j : M \times M \to M$ be projection on the $j$th factor, $j = 1, 2$. To say an operator $A$ on $L^2(E_i)$ is a smoothing operator means that there is a smooth section $k(x, y)$ of the bundle $\mathrm{Hom}(\pi_2^* E_i, \pi_1^* E_i)$ over $M \times M$, so that for all $s \in L^2(E_i)$.

$$(As)(x) = \int_M k(x,y)s(y)dy\,.$$

Note that $k(x,y)$ is a linear map from $E_{i,y}$, the fiber over $y$, to $E_{i,x}$, the fiber over $x$, so $k(x,x) : E_{i,x} \to E_{i,x}$ has a well defined trace. The section $k(x,y)$ is called the Schwartz kernel of $A$. Any smoothing operator on a compact manifold is of trace class and its trace is given by $\operatorname{tr}(A) = \int_M \operatorname{tr}\, k(x,x)dx$.

To see this for $e^{-t\Delta_i}$, note that its Schwartz kernel $k_t^i(x,y)$ must be given as follows: For each $\lambda_j$ choose on orthonormal basis $\phi_j^v$, $v = 1, \ldots, \dim E_i(\lambda_j)$ of $E_i(\lambda_j)$. Then

$$k_t^i(x,y) = \sum_{j=0}^{\infty} e^{-t\lambda_j}\left[\sum_v \phi_j^v(x)\phi_j^v(y)\right].$$

Here $k_t^i(x,y) : E_{i,y} \to E_{i,x}$ acts on $w \in E_{i,y}$ by

$$k_t^i(x,y)w = \sum_{j=0}^{\infty} e^{-t\lambda_j}\left[\sum_v \left(\phi_j^v(y), w\right)_i \cdot \phi_j^v(x)\right].$$

The trace of $k_t^i(x,x)$ is then given by

$$\sum_{j=0}^{\infty} e^{-t\lambda_j}\left[\sum_v \left(\phi_j^v(x), \phi_j^v(x)\right)_i\right]$$

and the result follows by integrating over $M$.

Now, since $e^{-t\lambda_0} = 1$ for all $t$, we have $e^{-t\lambda_0}\sum_{i=0}^{k}(-1)^i \dim E_i(\lambda_0) =$ Index$(E,d)$, for all $t$. In addition $e^{-t\lambda_j}\sum_{i=0}^{k}(-1)^i \dim E_i(\lambda_j) = 0$ for $j > 0$, and for all $t$. Thus we have

THEOREM 2.3. *If $M$ is compact, then for all $t > 0$,*

$$\begin{aligned}\operatorname{Index}(E,d) &= \sum_{j=0}^{\infty}\left[\sum_{i=0}^{k}(-1)^i e^{-t\lambda_j}\dim E_i(\lambda_j)\right]\\ &= \sum_{i=0}^{k}\left[\sum_{j=0}^{\infty}(-1)^i e^{-t\lambda_j}\dim E_i(\lambda_j)\right]\\ &= \sum_{i=0}^{k}(-1)^i \operatorname{tr}\, e^{-t\Delta_i}.\end{aligned}$$

The Index Theorem now follows from two other results.

1) Set $k_t(x) = \sum_{i=0}^{k}(-1)^i \mathrm{tr}\; k_t^i(x,x)$. Then for $t$ near 0, $k_t(x)$ has an asymptotic expansion of the form

$$k_t(x) = \sum_{j \geq -n} t^{j/2} a_j(x) .$$

As $\int_M k_t(x)dx = \sum_{i=0}^{k}(-1)^i \mathrm{tr}\; e^{-t\Delta_i} = \mathrm{Index}(E,d)$ is independent of $t$, we have

$$\mathrm{Index}(E,d) = \int_M a_0(x)dx .$$

Now, for any twisted Dirac operator $D_F^+$, one can prove that the differential $n$ form $a_0(x)dx$ is the degree $n$ part of the form constructed from the connections on $TM$ and $F$ which represents $\widehat{\mathcal{A}}(TM) \cdot ch(F)$. Thus, we have

$$\mathrm{Index}(D_F^+) = \int_M \widehat{\mathcal{A}}(TM) \cdot ch(F) .$$

2) Index$(E,d)$ depends only on the $K$ theory class $[\sigma(E,d)]$. Given this and the formula above for Index$(D_F^+)$, one may use well known arguments in $K$ theory to extend the result in 1. to all elliptic complexes. The essential fact is that the symbols of twisted Dirac operators generate the K theory with compact supports of $TM$ as an algebra over the K theory of $M$.

The difference between the formula in 1. and that in the Atiyah-Singer Index Theorem is accounted for by the fact that for the twisted Spin complex $(E^\pm \otimes F, D_F^+)$,

$$ch\big(\sigma(E^\pm \otimes F, D_F^+)\big) = ch(E^\pm, D^+) \cdot ch(F)$$

and

$$Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \mathbf{ch}\big(\sigma(E^\pm, D^+)\big) = \widehat{\mathcal{A}}(TM) .$$

For more on this see [ABP], [B], [G], [Gi], and [P].

## 3. The Lefschetz fixed point formula

### Endomorphisms of elliptic complexes

A collection $T = (T_0, \ldots, T_k)$ of complex linear maps $T_i : C^\infty(E_i) \to C^\infty(E_i)$ is an endomorphism of the complex $(E, d)$ provided

$$T_{i+1} \cdot d_i = d_i \cdot T_i$$

for all $i$. The $T_i$ then induce linear maps

$$T_i^* : H^i(E, d) \to H^i(E, d) .$$

When $M$ is compact, $H^i(E, d)$ is finite dimensional, and we may form $\operatorname{tr}(T_i^*)$. We define the Lefschetz number $L(T)$ of the endomorphism $T$ to be

$$L(T) = \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i^*) .$$

We are interested in the so called *geometric endomorphisms*. To define these, let $f : M \to M$ be a smooth map and for $i = 0, \ldots, k$, suppose that $A_i : f^* E_i \to E_i$ is a smooth bundle map. Then for each $x \in M$, we have a linear map

$$A_{i,x} : E_{i,f(x)} \to E_{i,x}$$

from the fiber of $E_i$ over $f(x)$, which is the fiber of $f^*E_i$ over $x$, to $E_{i,x}$ the fiber of $E_i$ over $x$. For any $s \in C^\infty(E_i)$, we define $T_i s \in C^\infty(E_i)$ by

$$(T_i s)(x) = A_{i,x} \cdot s\big(f(x)\big) .$$

We assume that the $A_i$ are chosen so that the $T_i$ define an endomorphism of $(E, d)$. For a geometric endomorphism associated to $f$, the Lefschetz number is denoted $L(f)$.

### Two Examples

1) The classic example is that of a smooth map acting on the de Rham complex. Then we have

$$\begin{aligned}
(E, d) &= \text{the de Rham complex of } M \\
f &= \text{an arbitrary map.} \\
A_i &= i\text{th exterior power of the adjoint, } df^*, \\
&\phantom{=} \text{ of the differential } df \text{ of } f, \text{ extended to } T^*_{\mathbf{C}} M \\
A_{i,x} &= \wedge^i df^*_x : \wedge^i T^*_{\mathbf{C}} M_{f(x)} \to \wedge^i T^*_{\mathbf{C}} M_x .
\end{aligned}$$

Then $T_i$ is the familiar $f_i^* : C^\infty(\wedge^i T_{\mathbf{C}}^* M) \to C^\infty(\wedge^i T_{\mathbf{C}}^* M)$

2) An action of a compact Lie group $G$ on a Spin manifold $M$ is an action of $G$ by orientation preserving isometries which preserves the given Spin structure on $M$. $G$ then acts on $E^+$ and $E^-$ and so also on $C^\infty(E^+)$ and $C^\infty(E^-)$ and this action commutes with $D^+$. Thus each $g \in G$ is an endomorphism of $(E^\pm, D^+)$ and $g \to L(g)$ defines a character of the group $G$. When $g = 1$, this is just the Spinor index of $M$.

Our aim is to relate the Lefschetz number of a geometric endomorphism to invariants defined on the fixed point set of $f$. To do so we need $f$ to be non-degenerate along its fixed point set in the sense that at each fixed point $p$, $df_p : TM_p \to TM_p$ has no eigen vectors with eigenvalue $+1$ in directions *transverse* to the fixed point set. Such fixed points are called non-degenerate. Note in particular that $f = id_M$ satisfies this condition !

THEOREM 3.1 (The Lefschetz Theorem, [AB], [AS]). *Let $f$ and $(E, d)$ be as above and $T$ a geometric endomorphism of $(E, d)$ for $f$. Assume that $M$ is compact and oriented and let $M_f$ be the fixed point set of $f$. Then $L(f)$ is given by an integral over $M_f$ of characteristic cohomology classes on $M_f$ determined by local data on $M_f$.*

The general formula for L(f) is quite complicated. We will give the formula for the Spin case (see [AH], p. 20). Let $G$ be as in 2. above acting on a compact Spin manifold $M$ of dimension $n = 2\ell$. Fix $g \in G$. The normal bundle $V_g$ of $M_g$ in $M$ has a canonical decomposition invariant under $g$,

$$V_g = \bigoplus_\lambda V_g(\lambda)$$

where $\lambda \in S^1 \subset \mathbf{C}$ and $g$ acts on $V_g(\lambda)$ by multiplication by $\lambda$. Only a finite number of $\lambda$ actually occur and we assume $\lambda = -1$ does not occur. (In applications it does not). Thus $V_g$ is a complex bundle and $V_g$ and $M_g$ are canonically oriented. For every complex number $z \neq 1$, set

$$Q_z(x) = z^{1/2} e^{-x/2} / (1 - z e^{-x}) .$$

Denote the associated multiplicative sequence by $B(\ ,z)$. Because of the factor of $z^{1/2}$, it is only defined up to sign. In [AH], p.21 it is explained how to remove this ambiguity. Then

$$L(g) = (-1)^\ell \int_{M_g} \widehat{\mathcal{A}}(TM_g) \cdot \prod_\lambda B\big(V_g(\lambda), \lambda\big) .$$

## OUTLINE OF THE PROOF OF THE LEFSCHETZ THEOREM

We outline a proof which does not rely in an essential way on the compactness of $M$. This allows us to generalize these results to complexes and endomorphisms defined along the leaves of a foliation of a compact manifold even though the leaves may be non-compact. A general reference for the material in this section is [RS].

We begin by redefining $e^{-t\Delta_i}$. Let $C$ be the curve in the complex plane

$$C = \{z = (x, y) \mid y^2 = x + 1\}$$

and set

$$e^{-t\Delta_i} = \frac{1}{2\pi i} \int_C \frac{e^{-t\lambda}}{(\lambda I - \Delta_i)} d\lambda \,,$$

i.e.

$$(e^{-t\Delta_i} s)(x) = \frac{1}{2\pi i} \int_C e^{-t\lambda} \left[(\lambda I - \Delta_i)^{-1} s\right](x)\, d\lambda$$

for $s \in L^2(E_i)$. A Riemannian manifold has bounded geometry if its curvature is bounded and its injectivity radius is bounded away from zero. On any complete manifold of bounded geometry, $(\lambda I - \Delta_i)^{-1}$ is a bounded operator on $L^2(E_i)$ for all $\lambda \in C$ so $e^{-t\Delta_i}$ is defined.

Note that when $M$ is compact, this agrees with our previous definition. To see this, use Cauchy's Theorem to show that the two definitions agree on an orthonormal basis.

## SOME FACTS ABOUT $e^{-t\Delta_i}$

Assume that $M$ is a complete manifold of bounded geometry. Then

1. As before, $e^{-t\Delta_i}$ is a smoothing operator with smooth Schwartz kernel $k_t^i(x, y)$ (ref. [S]), so if $M$ is compact, it is of trace class.
2. $\pi_{\ker \Delta_i}$, the projection onto the kernel of $\Delta_i$, is a smoothing operator, so if $M$ is compact, it is of trace class.
3. $\lim_{t \to \infty} e^{-t\Delta_i} = \pi_{\ker \Delta_i}$ in the strong operator topology, so if $M$ is compact, it follows that
$$\lim_{t \to \infty} \operatorname{tr}(e^{-t\Delta_i}) = \operatorname{tr}(\pi_{\ker \Delta_i})$$
4. Let $T_i$ be as in the Lefschetz Theorem. Then $T_i e^{-t\Delta_i}$ is a smoothing operator with Schwartz kernel
$$k_t^{T_i}(x, y) = A_{i,x} k_t^i\big(f(x), y\big) \,,$$

and if $M$ is compact it has trace

$$\operatorname{tr}(T_i \cdot e^{-t\Delta_i}) = \int_M \operatorname{tr}\big(k_t^{T_i}(x,x)\big)\,dx\,.$$

5. As $t \to 0$, if $x \neq y$, $k_t^i(x,y) \to 0$ to infinite order and this convergence is uniform in distance $(x,y)$. Roughly speaking, this is because as $t \to 0$, $e^{-t\Delta_i} \to I$, and its Schwartz kernel is converging to the distribution on $M \times M$ which on each $\{x\} \times M$ is the Dirac $\delta$ distribution at $x$. Thus if $f(x) \neq x$, we have

$$\lim_{t\to 0} \operatorname{tr}\big(k_t^{T_i}(x,x)\big) = \lim_{t\to 0} \operatorname{tr}\big(A_{i,x} k_t^i\big(f(x),x\big)\big) = 0\,,$$

and given $\varepsilon > 0$, this convergence is uniform for all $x$ with distance $\big(x,f(x)\big) > \varepsilon$.

Now suppose that M is compact and consider

$$A(t) = \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i \cdot e^{-t\Delta_i}) = \sum_{i=0}^{k} (-1)^i \int_M \operatorname{tr}\big(k_t^{T_i}(x,x)\big)\,dx\,.$$

By 5. above, $\lim_{t\to 0} A(t)$ can be computed by integrating only over a neighborhood of the fixed point set $M_f$ of $f$. This integration can be done using only *local* information about $(E,d)$, $f$ and $T_i$ on $M_f$. Thus $\lim_{t\to 0} A(t)$ gives the right hand side of the Lefschetz Theorem.

6. By 3. above and the fact that $\ker \Delta_i \simeq H^i(E,d)$ we have

$$\begin{aligned}
\lim_{t\to\infty} \operatorname{tr}(T_i e^{-t\Delta_i}) &= \operatorname{tr}(T_i \cdot \pi_{\ker \Delta_i}) \\
&= \operatorname{tr}(T_i \cdot \pi^2_{\ker \Delta_i}) \\
&= \operatorname{tr}(\pi_{\ker \Delta_i} \cdot T_i \cdot \pi_{\ker \Delta_i}) \\
&= \operatorname{tr}(T_i^*)\,.
\end{aligned}$$

Then $\lim_{t\to\infty} A(t)$ gives the left hand side of the Lefschetz Theorem, so to complete the proof of the Lefschetz Theorem we need only show :

THEOREM 3.2. *$A(t) = \sum_{i=0}^{k} (-1)^i \operatorname{tr}\ (T_i \cdot e^{-t\Delta_i})$ is independent of $t$.*

*Proof.* Set

$$\phi(\Delta_i) = e^{-t_1\Delta_i} - e^{-t_2\Delta_i} = \Delta_i \psi(\Delta_i)$$

where

$$\psi(x) = \frac{e^{-t_1 x} - e^{-t_2 x}}{x}.$$

Now formally we have

$$\sum_{i=0}^{k}(-1)^i \mathrm{tr}(T_i e^{-t_1\Delta_i}) - \sum_{i=0}^{k}(-1)^i \mathrm{tr}(T_i e^{-t_2\Delta_i})$$

$$= \sum_{i=0}^{k}(-1)^i \mathrm{tr}\big(T_i \phi(\Delta_i)\big)$$

$$= \sum_{i=0}^{k}(-1)^i \mathrm{tr}\big(T_i \Delta_i \psi(\Delta_i)\big)$$

$$= \sum_{i=1}^{k}(-1)^i \mathrm{tr}\big(T_i d_{i-1} d_i^* \psi(\Delta_i)\big) + \sum_{i=0}^{k-1}(-1)^i \mathrm{tr}\big(T_i d_{i+1}^* d_i \psi(\Delta_i)\big)$$

We now show that the first sum is the negative of the second.

$$\sum_{i=1}^{k}(-1)^i \mathrm{tr}\big(T_i d_{i-1} d_i^* \psi(\Delta_i)\big)$$

$$= \sum_{i=1}^{k}(-1)^i \mathrm{tr}\big(d_{i-1} T_{i-1} d_i^* \psi(\Delta_i)\big)$$

$$= \sum_{i=1}^{k}(-1)^i \mathrm{tr}\big(T_{i-1} d_i^* \psi(\Delta_i) d_{i-1}\big)$$

$$= \sum_{i=1}^{k}(-1)^i \mathrm{tr}(T_{i-1} d_i^* d_{i-1} \psi(\Delta_{i-1}))$$

$$= \sum_{i=0}^{k-1}(-1)^{i+1} \mathrm{tr}(T_i d_{i+1}^* d_i \psi(\Delta_i))$$

and done. Of course this manipulation is purely formal and must be justified as we are working with operators on infinite dimensional spaces and not on finite dimensional ones. For this, see [ABP] and [HL 1]. Note also that in order to have a Lefschetz Theorem for complete manifolds of bounded geometry, it is only necessary to find an appropriate trace for which the above results hold.

## 4. The Lefschetz Theorem for foliated manifolds

Let $M$ be a compact $m$ dimensional manifold and $F$ a dimension $n$ foliation on $M$. Then $F$ is an $n$ dimensional subbundle of $TM$ such that for any two sections $X$, $Y \in C^\infty(F)$, $[X,Y] \in C^\infty(F)$. The Frobenius Theorem says that for each $x \in M$, there is a neighborhood $U$ of $x$ and a diffeomorphism

$$\phi : \mathbf{R}^n \times \mathbf{R}^q \to U \qquad n+q=m$$

so that for all $z \in \mathbf{R}^n \times \mathbf{R}^q$.

$$d\phi(T\mathbf{R}^n_z) = F_{\phi(z)} .$$

Such a $(U,\phi)$ is called a foliation chart. Given $x \in \mathbf{R}^q$, the submanifold $\phi(\mathbf{R}^n \times \{x\})$ is called a plaque, and is denoted $P^U_x$. It is a local integral submanifold of $F$. The submanifold $\phi(\{0\} \times \mathbf{R}^q)$ is denoted $\mathbf{R}^q_U$ and is called the transverse submanifold of $(U,\phi)$.

A leaf $L$ of $F$ is a maximal integral (i.e. $TL_x = F_x$ for all $x \in L$) submanifold of $M$. Thus $\dim L = n$. The Frobenius Theorem implies that through each point $x$ in $M$, there passes a unique leaf, denoted $L_x$. Each leaf is a complete manifold of bounded geometry and the bounds are uniform for all leaves.

We now extend the Lefschetz Theorem for compact manifolds to a Lefschetz Theorem for foliations of a compact manifold. This is joint work with Connor Lazarov [HL 1]. In fact, we show how to improve the results of [HL 1] by removing the assumption that $F$ admits a transverse invariant metric. For a K-theory version of this result, see the thesis of M-T. Benameur, [Be].

Choose a smooth metric on $M$. This induces a smooth metric on each leaf $L$, and $L$ is complete with respect to this metric. Two different metrics on $M$ induce quasi-isometric metrics on $L$.

### Haefliger forms

Let $\{U_i\}$ be a finite cover of $M$ by foliation charts. For $x \in U_i$, denote its plaque by $P^i_x$. If $U_i \cap U_j \neq \varnothing$ we define a local diffeomorphism $f_{ij}$ from $\mathbf{R}^q_{U_i}$ (hereafter denoted $\mathbf{R}^q_i$) to $\mathbf{R}^q_j$ as follows:

$$f_{ij}(x) = y \text{ if and only if } P^i_x \cap P^j_y \neq \varnothing .$$

The $f_{ij}$ generate the holonomy pseudogroup, denoted $H$, which acts on the transversal space $T = \cup_i \mathbf{R}^q_i$. We may (and do) assume that the $\mathbf{R}^q_i$ are disjoint.

Recall the following construction due to Haefliger [Ha]. Let $\Omega_c^k(T)$ be the space of bounded measurable complex valued $k$ forms on $T$ with compact support. Denote by $\Omega_c^k(T/H)$ the quotient of $\Omega_c^k(T)$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$ where $h \in H$ and $\alpha \in \Omega_c^k(T)$ has support contained in the range of $h$. Give $\Omega_c^k(T/H)$ the quotient topology of the usual sup norm topology on $\Omega_c^k(T)$. Note that $\Omega_c^k(T/H)$ does not depend of the choice of cover used to define it.

Denote by $\Omega^{p+k}(M)$ the space of bounded measurable complex valued $p+k$ forms on $M$. As the bundle $TF$ is oriented, there is a continuous open surjective linear map,

$$\int_F : \Omega^{p+k}(M) \to \Omega_c^k(T/H).$$

It is given as follows. Let $\omega \in \Omega^{p+k}(M)$ and let $\{\psi_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$. Set $\omega_i = \psi_i\omega$. We may integrate $\omega_i$ along the fibers of the submersion $\pi_i : U_i \to \mathbf{R}_i^q$ to obtain $\overline{\omega}_i \in \Omega_c^k(\mathbf{R}_i^q)$. Define $\int_F \omega$ to be the class of $\Sigma\overline{\omega}_i$ in $\Omega_c^k(T/H)$. It is independent of the choices made in defining it.

### Differential complexes on $M$ elliptic along $F$

A differential complex on $M$ along $F$ consists of:

a) a finite collection of finite dimensional complex vector bundles $E_0, \dots, E_k$ over $M$

b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \to C^\infty(E_{i+1})$$

with $d_{i+1} \cdot d_i = 0$

c) each $d_i$ differentiates only in leaf directions.

For the sake of simplicity we assume that each $d_i$ is first order.

Each of the classical complexes mentioned above (de Rham, Dolbeault, Signature and Twisted Spin) gives a leafwise complex on $M$ provided that the leaves have the required structures and that these structures are coherent from leaf to leaf (i.e. come from a global structure on $M$). For example, in the twisted Spin case, we require that the Spin structure on the leaves comes from a principal Spin($n$) bundle $P$ over $M$ with $P \times_{\mathrm{Spin}(n)} \mathbf{R}^n \simeq TF$, and that the leafwise auxiliary twisting bundle come from a bundle over $M$.

For a fixed leaf $L$, denote $E_i\big|_L$ by $E_i^L$ and by $C_0^\infty(E_i^L)$ the space of smooth sections of $E_i^L$ with compact support. The operator $d_i$ induces one, denoted also by $d_i$,

$$d_i : C_0^\infty(E_i^L) \to C_0^\infty(E_{i+1}^L)$$

and on $L$ we have the complex

$$0 \to C_0^\infty(E_0^L) \xrightarrow{d_0} C_0^\infty(E_1^L) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C_0^\infty(E_k^L) \to 0\,.$$

We say that the complex $(E,d)$ is elliptic along $F$ provided that for each leaf $L$, the above complex is elliptic. We assume that $(E,d)$ is elliptic along $F$.

## $L^2$ COHOMOLOGY OF $(E,d)$

Choose a smooth Hermitian metric on each bundle $E_i$ *over M*. These induce metrics on each $E_i^L$ and these metrics are unique up to quasi-isometry. Using these metrics we construct $d_i^* : C_0^\infty(E_{i+1}^L) \to C_0^\infty(E_i^L)$ just as we did before. We then construct

$$\Delta_i^L : C_0^\infty(E_i^L) \to C_0^\infty(E_i^L)$$

and we extend $\Delta_i$ to

$$\Delta_i^L : L^2(E_i^L) \to L^2(E_i^L)$$

just as before.

DEFINITION 4.1. *The ith $L^2$ cohomology of $(E,d)$ along the leaf $L$, denoted $H_L^i(E,d)$ is*

$$H_L^i(E,d) = \ker \Delta_i^L\,.$$

*The ith $L^2$ cohomology of $(E,d)$ is denoted $H^i(E,d)$ and it assigns to each leaf $L$ the ith cohomology of $(E,d)$ along $L, H_L^i(E,d)$.*

## SOME FACTS

1. $H_L^i(E,d)$ consists of smooth sections and $\dim_{\mathbb{C}} H_L^i(E,d)$ may be infinite but is always countable.
2. $\pi_L^i$, the projection of $L^2(E_i^L)$ onto $H_L^i(E,d)$, is a smoothing operator (on $L$) with smooth Schwartz kernel $k_L^i(x,y)$.
3. $k_L^i(x,y)$ is measurable as a function of $L$ and bounded independently of $L$. In particular, tr $k_L^i(x,x)$ is a bounded measurable function on $M$ whose restriction to each leaf $L$ is smooth.
4. Because of 3. above, we may define the dimension of $H^i(E,d)$ to be the zero dimensional Haefliger form

$$\dim\big(H^i(E,d)\big) = \int_F \operatorname{tr}\big(k_L^i(x,x)\big)\,dx\,,$$

where for any leaf $L$ we denote the volume form obtained from the metric on $L$ by $dx$. We may also define the Euler class of $(E,d)$ as

$$\chi(E,d) = \sum_{i=0}^{k}(-1)^i \dim H^i(E,d)\,.$$

GEOMETRIC ENDOMORPHISMS

Let $f : M \to M$ be a diffeomorphism and assume that for each leaf $L$ of $F$, $f(L) \subset L$. For each $i$, let

$$A_i : f^*E_i \to E_i$$

be a smooth bundle map. We assume that $T_i : C^\infty(E_i) \to C^\infty(E_i)$ where $(T_i s)(x) = A_{i,x}s\big(f(x)\big)$ satisfies

$$T_i d_{i-1} = d_{i-1} T_{i-1}\,.$$

The $T_i$ then induce maps

$$T_i^L : C_0^\infty(E_i^L) \to C_0^\infty(E_i^L)$$

satisfying

$$T_i^L d_{i-1} = d_{i-1} T_{i-1}^L\,.$$

We call such a family $T = (T_0, \ldots, T_k)$ the geometric endomorphism of $(E,d)$ defined by $f$ and $A = (A_0, \ldots, A_k)$. The $T_i^L$ extend to uniformly bounded linear maps

$$T_i^L : L^2(E_i^L) \to L^2(E_i^L)\,.$$

LEFSCHETZ NUMBER OF A GEOMETRIC ENDOMORPHISM

Set $T_{i,L}^* = \pi_i^L \cdot T_i^L \cdot \pi_i^L$ and denote its Schwartz kernel by $k_L^{T_i^*}(x,y)$. Then $k_L^{T_i^*}(x,y)$ is globally bounded, smooth on $L \times L$, and measurable. Thus $\operatorname{tr}\big(k_L^{T_i^*}(x,x)\big)$ is a bounded measurable function on $M$ which is smooth on each leaf $L$. We define the Lefschetz class of the geometric endomorphism $T$ to be the Haefliger zero form

$$L(T) = \sum_{i=0}^{k}(-1)^i \int_F \operatorname{tr}\big(k_L^{T_i^*}(x,x)\big)\,dx\,.$$

For our Lefschetz Theorem we shall also need two restrictions on the fixed point set, $N$ of $f$. We require:

1. $N = \bigcup_\alpha N_\alpha$ is a finite disjoint union of closed, connected submanifolds $N_\alpha$, each transverse to $F$.
2. for each $x \in N \cap L = \bigcup_\alpha N_\alpha^L$ where $N_\alpha^L = N_\alpha \cap L$, $df_x$ has no eigen vector (in $TL_x$) with eigenvalue $+1$ in directions transverse (in $L$!) to $N_\alpha^L$.

Note in particular that $f = id_M$ satisfies these conditions.

FIXED POINT INDICES

Let $\{U_i\}$ and $\{\psi_i\}$ be as above. Suppose that for each $L$ and $\alpha$ we are given a differential form $a_\alpha^L$ defined on $N_\alpha^L$. We define the Haefliger form $\int_N a$ as

$$\int_N a = \sum_i \sum_{N_\alpha^L \cap P_x^i \neq \phi} \int_{N_\alpha^L \cap P_x^i} \psi_i a_\alpha^L .$$

Note that for any plaque $P_x^i$, only a finite number of $N_\alpha^L$ satisfy $N_\alpha^L \cap P_x^i \neq \phi$. As $\int_{N_\alpha^L \cap P_x^i} \psi_i a_\alpha^L$ is a differential form on the transversal $\mathbf{R}_i^q$ of $U_i$, we may also consider it as a Haefliger form for $F$. As above, it is not difficult to show that the Haefliger form $\int_N a$ does not depend on the choices made in defining it.

THEOREM 4.2 (The Lefschetz Theorem for Foliations [HL 1]). *Let $M$, $F$, $f$, $T$, $A$ and $(E, d)$ be as above. To each $N_\alpha^L \subset N$ we may associate a differential form $a_\alpha^L$ which depends only on local data on $N_\alpha^L$ so that*

$$L(T) = \int_N a .$$

The proof follows the outline given above for the classical case, done leafwise. There are some very formidable technical obstacles, but these can be overcome (see [HL 1]).

If $(E, d)$ is the de Rham, Dolbeault, Signature or Twisted Spin complex of $F$, and $f = id_M$, and $T = id$, then $a_j^L$ is the usual local integrand formula (computed on each leaf, not on $M$) given by the Atiyah-Singer Index Theorem. We thus have an index theorem for foliated manifolds for these operators.

(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of $F$, his theorem is related to ours as the $L^2$ covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of $M$ which has one leaf (namely $M$), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e. $f \neq id_M$, $T = f^*$, $a_j^L$ is the usual local integrand (computed on the fixed point set in each leaf, not in $M$) given by the Atiyah-Singer $G$ Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer $G$ Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

## 5. Group Actions and the Lefschetz Theorem

Let $F$ be an oriented $2k$ dimensional foliation of a compact, oriented, Riemannian manifold $M$. Assume that $F$ admits a $\mathrm{Spin}(2k)$ structure. That is, there is a principal $\mathrm{Spin}(2k)$ bundle $P$ over $M$ and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF .$$

We may then construct the bundles $E^{\pm} = P \times_{\mathrm{Spin}(2k)} \Delta^{\pm}$. The leafwise Dirac operator $D^+$ is constructed using the Riemannian structure on the leaves of $F$ which is induced from $M$.

Let $G$ be a compact, connected Lie group acting by isometries on $M$, taking each leaf of $F$ to itself. $G$ then acts on $TF$. We assume that $G$ also acts on $P$ (commuting with the action of $\mathrm{Spin}(2k)$) so that the induced action on $P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF$ is the given action on $TF$. $G$ then acts on the bundles $E^{\pm}$ and it commutes with the operator $D^+$, i.e. $G$ is a group of geometric endomorphisms of the complex $(E^{\pm}, D^+)$.

Recall the $\widehat{A}$ genus defined in Section 1.

Definition 5.1. *The $\widehat{A}$ genus of $F$ is the Haefliger zero form*

$$\widehat{A}(F) = \int_F \widehat{A}_{k/2}(TF) .$$

*In particular, if $k$ is odd, $\widehat{A}(F) = 0$.*

Note that we have defined $\widehat{A}(F)$ as the *zero th* order part of $\int_F \widehat{A}(TF)$. For an interpretation of the higher order terms of $\int_F \widehat{A}(TF)$, see [He].

The Lefschetz Theorem for Foliations applied to the case $f = id_M$, $T = id$ says that $\widehat{A}(F)$ is equal to the index of the leafwise Spin complex, which is just $L(I)$. The Connes Index Theorem [C] says that it is also equal to the index of the holonomy covering leafwise Spin complex.

We now prove the theorem of the introduction, namely

THEOREM 5.2 ([HL 2]). *Let $F$ be an oriented foliation of a compact oriented manifold $M$ and assume that $F$ admits a Spin structure. If a compact connected Lie group acts non-trivially on $M$ as a group of isometries taking each leaf of $F$ to itself and preserving the Spin structure on $F$, then the $\widehat{A}$ genus of $F$ is zero.*

As a corollary, we have the well known result of Atiyah and Hirzebruch.

THEOREM 5.3 ([AH]). *Let $M$ be a compact connected oriented manifold which admits a Spin structure. If a compact connected Lie group acts non-trivially on $M$, then $\widehat{A}(M) = \int_M \widehat{A}(TM)$ is zero.*

Of course, this theorem and its proof were the inspiration for Theorem 5.2.

Now let $G$ be a compact connected Lie group acting on $M$ by isometries taking each leaf of $F$ to itself and preserving the Spin structure on $F$. We quote two results from [HL 2] and refer the reader to that paper for the proofs. Note that in [HL 1] and [HL 2], we assume that $F$ admits a transverse invariant measure. A careful reading of those papers shows that in fact we may disregard the invariant transverse measure and consider the traces used as taking values in the Haefliger zero forms of $F$ and all the results remain valid. See the remarks on this in [HL 3].

LEMMA 5.4. *The fixed point set of the action of $G$ is a closed submanifold of $M$ which is transverse to $F$.*

THEOREM 5.5. *The Lefschetz number $L(g)$ is a continuous function on $G$.*

*Proof of Theorem 5.2.* We may assume $G = S^1 \subset \mathbf{C}$. Let $N$ be the fixed point set of $G$, $N_\alpha$ a connected component of $N$, $L$ a leaf of $F$ and $y \in N_\alpha \cap L$. The normal bundle to $N_\alpha \cap L$ in $L$ at $y$ can be written as $\oplus V_y^j$, where $V_y^j$ is a complex vector space and $z \in G$ acts on $V_y^j$ by multiplication by $z^{m_j}$ for some positive integer $m_j$. It follows that the $V^j$ are complex $G$ vector bundles on $N_\alpha \cap L$.

Now let $z \in \mathbf{C}$, $z \neq 1$ and consider the function $R(x,z) = 1/(1-ze^{-x})$. It can be written as a formal power series in $x$ whose coefficients are rational functions in $z$ having a pole only at $z=1$, and no pole at $z=\infty$. To see this, write

$$\begin{aligned}\frac{1}{1-ze^{-x}} = \sum_{k=0}^{\infty}(ze^{-x})^k = \sum_{k=0}^{\infty} z^k e^{-kx} &= (1+z+z^2+z^3+\cdots) \\ &- (z+2z^2+3z^3+\cdots)x \\ &+ (z+2^2z^2+3^2z^3+\cdots)x^2/2! \\ &- \cdots .\end{aligned}$$

Set $f_0(z) = 1+z+z^2+\cdots = 1/(1-z)$, and for $n \geq 1$, set $f_n(z) = \sum_{k=1}^{\infty} k^n z^k$. Then $(-1)^n f_n(z)/n!$ is the coefficient of $x^n$ in $R(x,z)$ and it is obvious that $f_{n+1}(z) = zf_n'(z)$. An induction argument then shows that $f_n(z)$ is a rational function of $z$ with a pole only at $z=1$ and no pole at $z=\infty$. By induction we also have that $z^{1/2}f_n(z)$ has a pole only at $z=1$ and, as it is $\mathcal{O}(z^{-1/2})$ at $z=\infty$, it has no pole at $z=\infty$.

Now for fixed $z \neq 1$, set $Q(x,z) = z^{1/2}e^{-x/2}R(x,z)$, which is a formal power series in $x$. Denote the corresponding multiplicative sequence by $B(\ ,z) = \big(B_0(\ ,z), B_1(\ ,z), \ldots\big)$.

Let $z \in G = S^1$ be a topological generator (i.e. $z$ generates a dense subgroup). Then the fixed point set of $z$ is $N$ and $z$ acts on $V^j$ by multiplication by $z^{m_j}$. Let $d_j$ be the complex dimension of $V^j$ and set

$$B(V^j, z) = B_{d_j}(V^j, z^{m_j}).$$

$B(V^j,z)$ is a cohomology class on $N_\alpha \cap L$ whose coefficients are rational functions of $z$ having poles only at roots of unity and no pole at $z=\infty$. Set

$$B(N_\alpha \cap L, z) = \prod_j B(V^j, z).$$

As $B(V^j,z)$ contains the factor $(z^{m_j d_j})^{1/2}$, $B(N_\alpha \cap L)$ contains the factor $(z^d)^{1/2}$, $d = \sum m_j d_j$, and so is defined only up to sign. The choice of sign is determined as in [AH], page 21.

The Riemannian connection on $TM$ over $N_\alpha \cap L$ preserves the bundles $V^j$ and is a complex connection on each $V^j$. Using this connection and the Riemannian connection on $T(N_\alpha \cap L)$, we may construct the differential form

$w_\alpha^L(z)$ on $N_\alpha \cap L$ which represents the cohomology class $\widehat{\mathcal{A}}(N_\alpha \cap L) B(N_\alpha \cap L, z)$. Then $w_\alpha^L(z)$ is the form $a_\alpha^L$ given in the foliation Lefschetz theorem for $z$ acting on the leafwise Spin complex, and it defines a smooth form $w_\alpha(z)$ on $N_\alpha$. Thus for $z \in S^1$, $z$ not a root of unity, we have

$$L(z) = \int_N w(z) = \sum_\alpha \int_{N_\alpha} w_\alpha(z) .$$

Now notice that the right side of this equation defines a function $A(F, z)$ on the complex plane with values in the Haefliger forms of $F$. Also note that $A(F, z)$ has poles only at roots of unity and no pole at $z = \infty$, since $w_\alpha(z)$ has poles only at roots of unity and no pole at $z = \infty$. Because of the factor of $(z^d)^{1/2}$, $A(F, 0) = 0$. For $z \in S^1$, $z$ not a root of unity, $A(F, z) = L(z)$. But $L(z)$ is defined for all $z \in S^1$ and by Theorem 5.5 it is continuous on $S^1$. Thus $A(F, z)$ has no poles at all. Since it is analytic and bounded, it is constant and hence is identically zero. Therefore $L(z) = 0$ for all $z \in S^1$, but $L(1) = \widehat{\mathcal{A}}(F)$ so we are done.

The compactness of $G$ is essential, as in [HL 2], we give an example of an infinite discrete group acting by leaf preserving isometries on a compact oriented foliated manifold $M, F$ and $G$ preserves a Spin structure on $F$. The foliation $F$ admits an invariant transverse measure which defines a map from the Haefliger zero forms of $F$ to $\mathbf{C}$. The image of $\widehat{\mathcal{A}}(F)$ under this map is non-zero, so $\widehat{\mathcal{A}}(F) \neq 0$ .

## 6. The Rigidity Theorem of Witten

In 1986, Witten [W] predicted rigidity theorems for the indices of certain elliptic operators on manifolds with $S^1$ actions. The genesis for Witten's conjecture was his study of the Dirac operator on the free loop space $\mathcal{L}M$ (an infinite dimensional manifold) of a Spin manifold $M$. $\mathcal{L}M$ admits a natural $S^1$ action whose fixed point set is diffeomorphic to $M$. The sequences of bundles $R(q)$ and $R'(q)$ described below were derived from the normal bundle of $M$ in $\mathcal{L}M$ and from the formal analogue on $\mathcal{L}M$ of the fixed point formula for the Dirac operator in the finite dimensional case.

Let $D : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic operator on a compact manifold $M$ and suppose $M$ admits an $S^1$ action preserving $D$. Then as noted above, Index$(D)$ is a virtual $S^1$ module and has a decomposition into a finite sum of irreducible complex one dimensional representations

$$\mathrm{Index}(D) = \sum a_m L^m$$

where $z \in S^1$ acts on $L^m$ by multiplication by $z^m$. $D$ is called *rigid* if all the $a_m$ for $m \neq 0$ are zero, i.e. if the representation $L^m$, $m \neq 0$ occurs in kernel $D$ with multiplicity $a$ then it occurs in cokernel $D$ with the same multiplicity $a$.

Denote by $S^k(T)$ and $\lambda^k(T)$ the $k$th symmetric and exterior powers of $T = TM$ and set

$$S_a(T) = \sum_{k=0}^{\infty} a^k S^k(T)$$

$$\lambda_a(T) = \sum_{k=0}^{\infty} a^k \lambda^k(T).$$

Let $R_n$ and $R'_n$ be the sequences of bundles defined by the formal power series

$$R(q) = \sum_{n=0}^{\infty} q^n R_n = \bigotimes_{\ell=1}^{\infty} \lambda_{q^\ell}(T) \bigotimes_{m=1}^{\infty} S_{q^m}(T)$$

$$R'(q) = \sum_{n=0}^{\infty} q^{n/2} R'_n = \bigotimes_{\ell=\frac{1}{2},\frac{3}{2},\cdots} \lambda_{q^\ell}(T) \bigotimes_{m=1}^{\infty} S_{q^m}(T)$$

Now suppose $M$ is a $2n$ dimensional compact Riemannian Spin manifold and denote by $D^+$ the Dirac operator of $M$. For each $n$ we may form the operators

$$D^+ \otimes (E^+ \oplus E^-) \otimes R_n \quad \text{and} \quad D^+ \otimes R'_n.$$

THEOREM 6.1. *These operators are rigid under any $S^1$ action on $M$ by isometries, i.e. the induced action on the index of any of these operators is the trivial action.*

This is the theorem conjectured by Witten and first proven by Taubes [T]. A beautiful proof of it appears in [BT].

Roughly speaking Bott and Taubes' proof goes as follows. First they show that the Signature operator $d_S = D^+ \otimes (E^+ \oplus E^-)$ is rigid by an argument similar to that presented above. Combining this result with the power series $R(q)$ and interpreting $\mathrm{ch}(\mathrm{Index}\big(d_S \otimes R(q)\big)$ as a meromorphic function on the complex torus $T_{q^2} = \mathbf{C}^*/q^2$, they show that it has poles only at roots of unity and no poles on a certain circle $S^1 \subset T_{q^2}$. The Spin hypothesis then implies that it has no poles at all and hence is constant. Thus the character of $S^1$ given by its action on Index$(d_S \otimes R(q))$ is constant, and so the action must be trivial as claimed. They then give separate arguments to extend this result to $D^+ \otimes R'_n$.

These results all extend in a straight forward way to $S^1$ actions preserving a foliation (see [HL 2]).

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*(Reçu le 10 décembre 1995)*

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