4. Manifolds That Admit Chaotic Group Actions

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(b) *If* $G \leqslant H \leqslant \mathrm{Hom}(M)$ *and* G *has finite index in* H *and if the action of* G *on* M *is chaotic, then the action of* H *on* M *is chaotic.*

(c) *If* M *is locally compact and if* $\mathrm{Hom}(M)$ *is given the compact-open topology, then the action of* G *on* M *is chaotic if and only if the action on* M *of the closure* $\bar{G}$ *of* G *in* $\mathrm{Hom}(M)$ *is chaotic.*

Proof. In Part (a), notice that if a point $x \in M$ has finite orbit under K, then x obviously has finite orbit under H. So if the action of K has finite orbits dense, then the action of H has finite orbits dense. On the other hand, if the action of G is topologically transitive, then clearly the action of H is also topologically transitive. So Part (a) holds. Part (b) is similar to Part (a).

In Part (c), again if the action of $\bar{G}$ has finite orbits dense, then the action of G has finite orbits dense. Now suppose that the action of $\bar{G}$ is topologically transitive. Let U and V be two non-empty open subsets of M. Then there exists $g \in \bar{G}$ such that $g(U) \cap V$ is non-empty. Let x be an element of $U \cap g^{-1}(V)$ and let Θ be the open subset of $\bar{G}$ composed of elements that send x into V. Then $g \in \Theta$ and since G is dense in $\bar{G}$, there exists $h \in G \cap \Theta$. So $h(U) \cap V$ is non-empty and hence the action of G is topologically transitive.

Conversely, if M is locally compact, then the natural map $\mathrm{Hom}(M) \times M \to M$ is continuous. So, if a point $x \in M$ has finite orbit under G, then since G is dense in $\bar{G}$, one has that $G(x)$ is dense in $\bar{G}(x)$. Hence $\bar{G}(x)$ is finite. So if the action of G has finite orbits dense, then the action of $\bar{G}$ has finite orbits dense. Finally, if the action of G is topologically transitive, then obviously so too is the action of $\bar{G}$. $\square$

4. Manifolds That Admit Chaotic Group Actions

Chaotic homeomorphisms of the 2-dimensional disc can be constructed as follows. Starting with any Anosov diffeomorphism of the torus $\mathbf{T}^2$, one can quotient by the map $\sigma : x \mapsto -x$, to obtain a chaotic homeomorphism on the sphere $\mathbf{S}^2$. (This map was used in [Wa], p. 140 to show that expansiveness is not preserved under semi-conjugation.) Then, by blowing up the origin to a circle, one obtains a chaotic homeomorphism on the closed disc. Unfortunately this latter homeomorphism is not the identity on the boundary. This can be rectified by making a slight modification of the above construction. Instead of starting with an Anosov diffeomorphism of $\mathbf{T}^2$, one starts with linked twist map [D1] of the torus $\mathbf{T}^2$. A linked twist map is an appropriately chosen composition of Dehn twists. Consider the particular linked twist map $\bar{f}$ defined as follows: by representing $\mathbf{T}^2$ as the square with vertices $(\pm 1/2, \pm 1/2)$, and edges identified in the usual manner, consider the maps $g: \mathbf{T}^2 \to \mathbf{T}^2$ and $h: \mathbf{T}^2 \to \mathbf{T}^2$ defined by

$$g(x, y) = \begin{cases} (x, y + 2x + 1/2) & \text{if } |x| \leqslant 1/4 \\ (x, y) & \text{otherwise,} \end{cases}$$

$$h(x, y) = \begin{cases} (x + 2y + 1/2, y) & \text{if } |y| \leqslant 1/4 \\ (x, y) & \text{otherwise.} \end{cases}$$

Then set $\bar{f} = g \circ h$. By [D1], the map $\bar{f}$ is chaotic on the set

$$M = \{(x, y) : |x| \leqslant 1/4 \text{ or } |y| \leqslant 1/4\} .$$

Moreover, $\bar{f}$ is the identity on the boundary of M. Now, quotienting by the map $\sigma: (x, y) \mapsto (-x, -y)$, one obtains a chaotic homeomorphism f on the disc D^2 and by construction f is also the identity on the boundary.

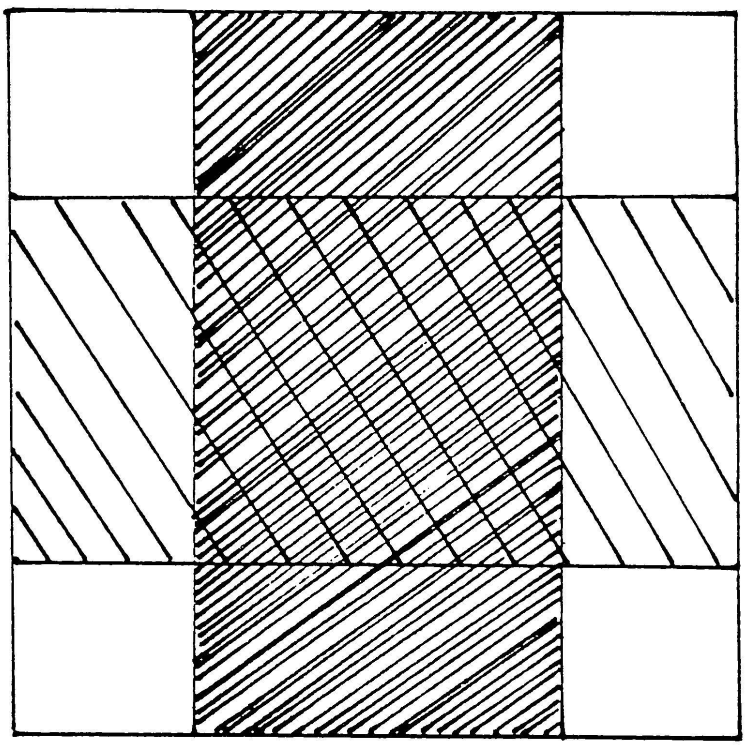

FIGURE 1

Using the above map f, one can clearly obtain chaotic homeomorphisms on all closed surfaces (orientable or not); one simple constructs the surface by identifying boundary arcs on the disc in the standard manner and then obtains the required homeomorphism from f, by semi-conjugacy.

THEOREM 3. *Every compact surface (with or without boundary) admits a chaotic* $\mathbf{Z}$*-action (that is, a chaotic homeomorphism).*

Now it is folkloric that the circle admits no invertible chaotic dynamical system. Indeed, we prove that no group acts chaotically on the circle. In fact, one has:

THEOREM 4. *No infinite group acts faithfully with finite orbits dense on the circle* $\mathbf{S}^1$.

Proof. It is well known and easy to prove that $\mathbf{S}^1$ admits no chaotic homeomorphism (see for example [Si]). In fact, one has the following elementary Lemma, which we give without proof:

LEMMA. *Suppose that* ϕ *is a orientation preserving homeomorphism of the circle* $\mathbf{S}^1$ *having dense periodic points. If* ϕ *has a fixed point, then* ϕ *is the identity.*

Now, returning to Theorem 4, suppose that a group G acts faithfully with finite orbits dense on $\mathbf{S}^1$. Then the elements of G all have dense periodic points. Let $x \in \mathbf{S}^1$ be a point with finite orbit under the action of G. Now let G_x^+ be the subgroup of G comprised of the orientation preserving elements that fix x. By the above Lemma, G_x^+ consists only of the identity map. Hence, since G_x^+ is a subgroup of finite index in G, we have that G is finite. □

By the classical theory of S. Cairns and J. Whitehead (see [KiSi]), every smooth compact manifold is triangulable and consequently can be constructed from the closed ball by identification of simplices in its boundary. Given the proof of Theorem 3 above, the obvious question is:

QUESTION 1. *Is there a chaotic homeomorphism of the closed 3-ball* B^3 *which is the identity on the boundary?*

The method used in dimension 2 doesn't seem to generalize to dimension 3. The 3-ball can be obtained by considering the action of $\mathbf{Z}_2 \times \mathbf{Z}_2$ on $\mathbf{T}^3$, by rotations through π about the x, y and z axes. However, the linked twist maps on $\mathbf{T}^3$ are not respected by this action. The ideas in [BFK] may be useful here; this paper shows that every compact manifold of dimension greater than one admits a Bernoulli diffeomorphism. (Bernoulli diffeomorphisms are ergodic and hence transitive, but they do not all have dense periodic points.)

Finally, as promised in the introduction, we give the:

Example. The group $G = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ acts faithfully and chaotically on $\mathbf{T}^2$ in such a way that none of the elements of G act chaotically on $\mathbf{T}^2$ in R. Devaney's sense.

Proof. First, as described above, there exist chaotic homeomorphisms of the closed disc (and hence of the closed square) which are the identity on the boundary. Let f be such a homeomorphism. Now consider $\mathbf{T}^2$ as the unit square with vertices (i, j) with $i, j \in \{0, 1\}$ and with edges identified in the usual manner; that is $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. Now use the x and y axes to subdivide $\mathbf{T}^2$ into 4 isometric subsquares. Let F be the homeomorphism of $\mathbf{T}^2$ obtained by applying f in each of the 4 subsquares. Let $g: \mathbf{T}^2 \to \mathbf{T}^2$ be the translation $g(x, y) = (x + 1/2, y)$. Similarly, define h by $h(x, y) = (x, y + 1/2)$. Then the group $G = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ generated by F, g and h acts chaotically on $\mathbf{T}^2$. But clearly G contains no element which acts chaotically on $\mathbf{T}^2$.

5. Other Questions

In this section we present some open questions which we have been unable to resolve. The main question is the following:

QUESTION 2. *Is there a faithful chaotic action of* $\mathbf{Z} \times \mathbf{Z}$ *on the torus* $\mathbf{T}^2$ *or the sphere* $\mathbf{S}^2$*?*

This question is of interest since in order to further the study of chaotic actions, one would naturally look to actions, on low dimensional manifolds, of groups which are simple generalizations of $\mathbf{Z}$. Because of Theorem 4, the obvious place to start is in dimension 2. Now the group $\mathbf{Z} \times \mathbf{Z} (= \mathbf{Z}^2)$ acts chaotically on $\mathbf{T}^4$. But it is not clear whether $\mathbf{Z}^2$ acts chaotically and faithfully on $\mathbf{T}^2$. Notice that $SL(2, \mathbf{Z})$ has no subgroup isomorphic to $\mathbf{Z}^2$. Indeed, $PSL(2, \mathbf{Z})$ is a free product $\mathbf{Z}_2 * \mathbf{Z}_3$ (see [MKS]) and hence by Kurosh's theorem (see [LS]), it cannot have $\mathbf{Z}^2$ as a subgroup. But $PSL(2, \mathbf{Z})$ is the quotient of $SL(2, \mathbf{Z})$ by the group $\{\pm \mathrm{Id}\} \cong \mathbf{Z}_2$. So $SL(2, \mathbf{Z})$ cannot have $\mathbf{Z}^2$ as a subgroup either.

It follows from the above discussion that if $G = \mathbf{Z}^2$ acts chaotically and faithfully on $\mathbf{T}^2$, then G cannot contain a linear hyperbolic toral automorphism. Indeed, according to [AdPa], if f is a linear hyperbolic toral automorphism and if g is a homeomorphism of $\mathbf{T}^n$ which commutes with f, then g is also a linear toral automorphism. (For more on commuting diffeomorphisms of tori, see [KaSp].)