Introduction

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CUBIC FORMS AND COMPLEX 3-FOLDS

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INTRODUCTION

Nowadays, complex or algebraic manifolds are classified by Kodaira dimension. This classification is natural and fruitful, but in the complex case another point of view is possible. In this approach one starts with a topological or differentiable manifold X and asks for all complex or algebraic structures on X. Though this more traditional way of thinking can't replace the classification by Kodaira dimension, it remains useful and attractive and it has led to a number of wellknown if not famous problems. It suffices to recall Severi's problem: find all complex structures on $\mathbf{P}^2$, considered as a topological 4-manifold, or the same question asked for $S^2 \times S^2$ seen either as a topological or a differentiable manifold. For complex dimension 2 the work of Freedman on the topology of 4-folds as well as the work of Donaldson and many of his followers of course put this point of view very much at the centre of attention [O/V], [F/M].

In the past decades progress on the Kodaira classification for dimension 3 has been enormous ([Mo], [K/M/M], [Ko1]), but the same can't be said about the relations between the topological and differentiable structures of 6-manifolds and the complex or algebraic structures they admit.

Let us restrict ourselves to the simplest case, the case of compact, oriented, simply-connected 6-manifolds without torsion. Their topological classification was carried out by Wall and Jupp ([W], [J]), who also determined which of them admit a differentiable structure, and for these showed that the differentiable classification coincides with the topological classification. This does not hold for the homotopy classification; in many cases there are even infinitely many homeomorphism classes of one and the same homotopy type. Apart of course from Stiefel-Whitney classes, Pontrjagin class and triangulation class the essential invariant is the cup form $H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to H^6(X, \mathbf{Z})\ (\cong \mathbf{Z})$. It is not difficult to characterize those forms which arise as cup forms of a 6-fold in question (below), but it remains very difficult to classify cubic forms up to $GL(\mathbf{Z})$-equivalence. Relatively few results are known in this direction, even for the lowest ranks.

The corresponding 4-folds are the simply connected ones, i.e. the 4-folds occurring in the work of Freedman and most of the papers of the Donaldson school. Here the crucial invariant is a unimodular form on $H^2(X, \mathbf{Z})$, namely the cup form $H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to H^4(X, \mathbf{Z})$. For differentiable manifolds this form completely determines the homeomorphism type (this also holds in the topological case if the cupform is even, whereas for odd forms there are two homeomorphism types), but by no means for the diffeomorphism type. So considering the relation between the homotopy, the topological and the differential classification there is a big difference between dimensions 4 and 6. The next question: which topological 4-folds carry a complex structure, is equivalent to asking which unimodular, $\mathbf{Z}$-valued symmetric bilinear forms are realisable by complex or algebraic surfaces.

It is related to the well-known inequality $c_1^2 \leqslant 3c_2$ and has been solved to a considerable extent.

Though in the case of 6-folds the corresponding question about the realisability of cubic forms is definitely weaker than the question which 6-folds carry a complex or algebraic structure, it still remains of much interest. In the second half of this paper we say something about algebra and arithmetic of cubic forms and consider the apparently largely untouched question of the realisability of *complex* forms by complex manifolds. Apart from a considerable number of examples some conditions for Kähler manifolds are given. And to show how few 6-folds of the type in question actually carry Kähler structures, we add a theorem about Kähler structures on the set of 6-folds with $b_2 = 1$, $b_3 \leqslant$ constant and $w_2 \neq 0$.

The first part of this paper surveys the results of Wall and Jupp referred to before, and deals with the homotopy classification. By putting together (for the first time?) all this in a rather systematic way we hope to contribute to the knowledge of complex 3-folds from a topological point of view.

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1. Topological classification of certain 6-manifolds

The topological classification of 1-connected, closed, oriented, 6-dimensional manifolds has been developped in a sequence of papers by C.T.C. Wall [W], P. Jupp [J], and A. Žubr [Z1], [Z2], [Z3]. Roughly speaking, their main result is that the topological classification of these 6-manifolds is equivalent to the arithmetic classification of certain systems of invariants naturally associated with them.

The aim of this section is to review these results and to reformulate the arithmetic classification problem in a way which makes it accessible to further investigation.

1.1 Homeomorphism types and C^∞-structures

Let X be a closed, oriented, 6-dimensional topological manifold; we assume that X is 1-connected with torsion-free homology. The *basic invariants* of X are [J]: