

# 5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

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*Proof.* Since  $n > 1$ , if  $T \subset T^n$  is a circle subgroup then  $\chi(X/T) = 0$ . Applying Theorem 4.5 to the bundle  $T \rightarrow X \rightarrow X/T$  yields the conclusion.  $\square$

COROLLARY 4.8. *If  $n > 1$  then  $\chi_1(T^n): \mathbf{Z}^n \rightarrow \mathbf{Z}^n$  is zero.*  $\square$

## 5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

Let  $G$  be a group of type  $\mathcal{F}$ . Gottlieb's theorem (see Propositions 1.3 and 2.4) asserts that if  $\chi(G) \neq 0$  then  $Z(G)$ , the center of  $G$ , is trivial. We prove an analogous theorem for  $\chi_1(G; \mathbf{Q})$ : if  $\chi_1(G; \mathbf{Q}) \neq 0$  then the center of  $G$  is infinite cyclic provided  $G$  satisfies an extra hypothesis (explained below) related to the Bass Conjecture; see Proposition 5.2 and Theorem 5.4.

Throughout this section  $R$  will be a commutative ground ring. Let  $S$  be any associative  $R$ -algebra with unit. The Hochschild homology group  $HH_0(S)$  is the  $R$ -module  $S/[S, S]$  where  $[S, S]$  is the  $R$ -submodule of  $S$  generated by  $\{ab - ba \mid a, b \in S\}$ ; see §2. Recall that  $K_0(S)$  is the abelian group  $F/A$  where  $F$  is the free abelian group generated by the set of all isomorphism classes  $[M]$  of finitely generated projective right  $S$ -modules  $M \subset \bigoplus_{i=1}^{\infty} S$  and  $A$  is the subgroup of  $F$  generated by relations of the form  $[M_1 \oplus M_2] - [M_1] - [M_2]$ . Since a finitely generated projective module is the image of a finitely generated free module under an idempotent homomorphism, each element of  $K_0(S)$  can be represented by an idempotent matrix over  $S$ . The *Hattori-Stallings* trace  $T_0: K_0(S) \rightarrow HH_0(S)$  is defined as follows. Let  $A: M \rightarrow M$  be an idempotent endomorphism of a free, finitely generated right  $S$ -module  $M$  representing  $x \in K_0(S)$ . If  $[A]$  is the matrix of  $A$  with respect to a given basis for  $M$  then  $T_0(x)$  is defined to be  $T_0([A]) \in HH_0(S)$ .

Consider the groupring,  $RG$ , of a group  $G$  over  $R$ . Then  $HH_0(RG)$  is naturally isomorphic to the free  $R$ -module generated by  $G_1$ , the set of conjugacy classes of  $G$  (see §2 for an explanation in the case  $R = \mathbf{Z}$ ). Recall that for  $g \in G$  we write  $C(g) \in G_1$  for the conjugacy class of  $g$ ,  $HH_0(RG)_{C(g)}$  for the summand of  $HH_0(RG)$  corresponding to  $C(g)$  and  $x_{C(g)}$  for the  $C(g)$ -component of  $x \in HH_0(RG)$ . Also write  $HH_0(RG) = HH_0(RG)_{C(1)} \oplus HH_0(RG)'$  where  $1 \in G$  is the identity element of  $G$ , and  $HH_0(RG)'$  is the direct sum of the remaining summands. The augmentation homomorphism  $\varepsilon: RG \rightarrow R$  induces a homomorphism  $\varepsilon_*: HH_0(RG) \rightarrow HH_0(R) = R$ .

**STRONG BASS PROPERTY.** We say that the group  $G$  has the *Strong Bass Property over  $R$* , abbreviated to “SBP over  $R$ ”, if the image of the homomorphism  $T_0: K_0(RG) \rightarrow HH_0(RG)$  lies in the  $HH_0(RG)_{C(1)}$  summand.

**WEAK BASS PROPERTY.** We say that the group  $G$  has the *Weak Bass Property over  $R$* , abbreviated to “WBP over  $R$ ”, if the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)' \xrightarrow{\varepsilon_*} R$$

is zero.

Clearly, if  $G$  has the SBP over  $R$  then it also has WBP over  $R$ . There are well-known conjectures concerning the SBP and the WBP (see [Bass], [DV] and [St, §4.1]):

**STRONG BASS CONJECTURE.** Every group has the SBP over  $\mathbf{Z}$ .

**WEAK BASS CONJECTURE.** Every group has the WBP over  $\mathbf{Z}$ .

The corresponding conjectures are false over  $\mathbf{Q}$  for a group which has nontrivial torsion; instead, one could conjecture:

**STRONG BASS CONJECTURE OVER  $\mathbf{Q}$ .** Every torsion free group has the SBP over  $\mathbf{Q}$ .

**WEAK BASS CONJECTURE OVER  $\mathbf{Q}$ .** Every torsion free group has the WBP over  $\mathbf{Q}$ .

Each element of the center of  $G$ ,  $Z(G)$ , makes up its own conjugacy class. Given a subgroup  $N$  of  $Z(G)$ , let  $HH_0(RG)_N = \bigoplus_{C(g) \in c(N)} HH_0(RG)_{C(g)}$  where  $c(N)$  is the set of conjugacy classes in  $G$  represented by elements of  $N$ . Then  $HH_0(RG) = HH_0(RG)_N \oplus HH_0(RG)'_N$  where  $HH_0(RG)'_N$  is the direct sum of the summands corresponding to the conjugacy classes not in  $c(N)$ .

**PROPERTY C.** We say that the group  $G$  has *Property C over  $R$*  if there exists a non-empty subset  $N$  of  $Z(G)$  such that the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)'_N \xrightarrow{\varepsilon_*} R$$

is zero.

By taking  $N$  to be the trivial subgroup of  $Z(G)$  we see that if  $G$  has the WBP over  $R$  then it also has Property C over  $R$ .

Recall that a group  $G$  is said to have finite cohomological dimension over the commutative ground ring  $R$  if there exists an integer  $N$  such that  $H^k(G, M) = 0$  for all  $RG$ -modules  $M$  and for all  $k > N$ . Also,  $G$  is said to be of type  $FP_\infty$  over  $R$  if the trivial  $RG$ -module  $R$  has a resolution by finitely generated projective  $RG$ -modules.

The following proposition is derived from the techniques of [St, §3].

**PROPOSITION 5.1.** *Let  $R$  be a principal ideal domain of characteristic  $p \geq 0$ . Suppose that  $G$  is of type  $FP_\infty$  over  $R$  and has finite cohomological dimension over  $R$ . Suppose also that  $G$  has a subgroup  $H$  of finite index which has Property C over  $R$ ; furthermore, if  $p > 0$  assume that  $p$  does not divide  $[G:H]$ . If the Euler characteristic  $\chi(G; R) \equiv \sum_{i \geq 0} (-1)^i \text{rank}_R H_i(G, R)$  is non-zero modulo  $p$  then the center of  $G$  is finite.*

*Proof.* Since  $H$  is of finite index in  $G$ ,  $H$  is also of type  $FP_\infty$  over  $R$  ([Bi, Proposition 2.5]) and has finite cohomological dimension over  $R$  ([Bi, Corollary 5.10]). Furthermore,  $\chi(H; R) = [G:H] \chi(G; R)$  and so  $\chi(H; R) \not\equiv 0 \pmod{p}$ .

We show that the center of  $H$ ,  $Z(H)$ , is finite. It then follows that the center of  $G$ ,  $Z(G)$ , is finite because there is an exact sequence  $1 \rightarrow Z(G) \cap H \rightarrow Z(G) \rightarrow N_G(H)/H$ , where  $N_G(H)$  is the normalizer of  $H$  in  $G$ , and the groups  $N_G(H)/H$  and  $Z(G) \cap H \subset Z(H)$  are finite.

Since  $H$  is of type  $FP_\infty$  over  $R$  and has finite cohomological dimension over  $R$ , it follows that  $R$  has a finite resolution,  $0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow R \rightarrow 0$ , where each  $P_j$  is a finitely generated projective  $RH$ -module (combine [Bi, Proposition 4.1(b)] and [Bi, Proposition 1.5]). Let  $\varepsilon: RH \rightarrow R$  be the augmentation homomorphism. Consider the commutative square:

$$\begin{array}{ccc} K_0(RH) & \xrightarrow{T_0} & HH_0(RH) \\ \varepsilon_* \downarrow & & \varepsilon_* \downarrow \\ K_0(R) & \xrightarrow{T_0} & HH_0(R) \cong R \end{array}$$

Let  $\alpha = \sum_{n \geq 0} (-1)^n [P_n] \in K_0(RH)$ . Then  $\varepsilon_*(T_0(\alpha)) = T_0(\varepsilon_*(\alpha)) = \chi(H; R) \cdot 1$  where  $1 \in R$  is the unity in  $R$ . The second equality is the classical Hopf trace formula over the principal ideal domain  $R$ . (Stallings ([St]) calls  $T_0(\alpha) \in HH_0(RH)$  the *Euler characteristic* of the projective  $RH$ -complex  $P_*$ .) Since  $H$  is assumed to have Property C over  $R$ , there is a non-empty subset  $N$  of  $Z(H)$  such that  $\varepsilon_*(T_0(\alpha)) = \varepsilon_*(T_0(\alpha)_N)$ .

Since  $\chi(H; R) \neq 0 \pmod p$ , it follows that  $T_0(\alpha)_{C(h)} \neq 0$  for some  $h \in N \subset Z(H)$ . Recall that the group  $Z(H)$  acts on  $HH_0(RH)$  by  $(rC(h))\omega = rC(h\omega^{-1})$  where  $r \in R$ ,  $h \in H$ , and  $\omega \in Z(H)$ . By [St, Theorem 3.4] (compare (2.3) above),  $T_0(\alpha)\omega = T_0(\alpha)$  for all  $\omega \in Z(H)$ . Since an element of  $HH_0(RH)$  is a *finite* linear combination of conjugacy classes, it follows that the condition  $T_0(\alpha)_{C(h)} \neq 0$  with  $h$  as above is impossible unless  $Z(H)$  is finite.  $\square$

We will be interested in groups with the property that certain of their central quotients have Property C “virtually”:

**PROPERTY D.** Let  $p \geq 0$  be the characteristic of  $R$ . We say that the group  $G$  has *Property D over  $R$*  if the following condition holds. Given any element  $\tau$  in the center of  $G$  with the property that the extension class  $e_R \in H^2(G/\langle \tau \rangle; R)$  is zero (where  $\langle \tau \rangle$  is the cyclic subgroup generated by  $\tau$ ), there is a finite index subgroup  $H \subset G/\langle \tau \rangle$  such that  $H$  has Property C over  $R$ ; moreover, if  $p > 0$  we require that  $p$  does not divide  $[G : H]$ .

The next Proposition is our “higher” analog of Gottlieb’s theorem over a field of arbitrary characteristic; Theorem 5.4, below, is a more usable version over  $\mathbf{Q}$ .

**PROPOSITION 5.2.** *Let  $\mathbf{F}$  be a field. Suppose  $G$  is a group of type  $\mathcal{F}$  such that  $G$  has Property D over  $\mathbf{F}$ . If  $\chi_1(G; \mathbf{F}) \neq 0$ , then the center of  $G$  is infinite cyclic.*

*Proof.* Let  $\tau$  be any element in  $Z(G)$ , the center of  $G$ , such that  $\chi_1(G; \mathbf{F})(\tau) \neq 0$ . Since  $G$  is necessarily torsion free, the group  $T = \langle \tau \rangle$  is infinite cyclic. By [Bi, Proposition 2.7]  $G/T$  is of type  $FP_\infty$  over  $\mathbf{Z}$  (and hence over any commutative ring). Since  $T$  is central, the Serre fibration  $S^1 \simeq K(T, 1) \rightarrow K(G, 1) \rightarrow K(G/T, 1)$  is orientable. By Theorem 4.2,  $e_{\mathbf{F}} = 0 \in H^2(G/T; \mathbf{F})$ , and  $\chi(G/T; \mathbf{F})$  exists and is non-zero mod  $p$  where  $p \geq 0$  is the characteristic of  $\mathbf{F}$ . Consider the following portion of the cohomology Gysin sequence of the fibration  $S^1 \rightarrow K(G, 1) \rightarrow K(G/T, 1)$ , with coefficients in an arbitrary  $\mathbf{F}G/T$ -module  $M$ :

$$H^{i-2}(G/T; M) \xrightarrow{\cup e_{\mathbf{F}}} H^i(G/T; M) \rightarrow H^i(G; M).$$

Since  $e_{\mathbf{F}} = 0$ ,  $H^i(G/T; M) \rightarrow H^i(G; M)$  is injective and so  $H^i(G/T, M) = 0$  for  $i > \dim X$  where  $X$  is a finite complex homotopy equivalent to  $K(G, 1)$ . In particular, Proposition 5.1 applies to  $G/T$  and so the center of  $G/T$  is

finite. Since the image of  $Z(G)$  in  $G/T$  is central, it follows that  $Z(G)$  is an extension of  $T$  by a finite group. Thus  $Z(G)$  is infinite cyclic since  $G$  is torsion free.  $\square$

Property D may be hard to verify for an arbitrary coefficient ring  $R$ . However, when  $R = \mathbf{Q}$  we have:

**PROPOSITION 5.3.** *Let  $G$  be a finitely generated group which has the WBP over  $\mathbf{Q}$ . Then  $G$  has Property D over  $\mathbf{Q}$ .*

*Proof.* Suppose  $\tau \in Z(G)$  is such that the extension class  $e_{\mathbf{Q}} \in H^2(G/T; \mathbf{Q})$  is zero where  $T$  is the cyclic subgroup of  $G$  generated by  $\tau$ . Consider the following portion of the long exact sequence in cohomology associated to the short exact sequence of coefficients,  $0 \rightarrow \mathbf{Z} \xrightarrow{j} \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$ :

$$H^1(G/T; \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta} H^2(G/T; \mathbf{Z}) \xrightarrow{j_*} H^2(G/T; \mathbf{Q}).$$

By exactness,  $j_*(e_{\mathbf{Z}}) = e_{\mathbf{Q}} = 0$  implies  $e_{\mathbf{Z}} = \delta(u)$  for some  $u \in H^1(G/T, \mathbf{Q}/\mathbf{Z})$ . Let  $H = \ker(u)$  where we regard  $u$  as an element of  $\text{Hom}(G/T, \mathbf{Q}/\mathbf{Z}) \cong H^1(G/T, \mathbf{Q}/\mathbf{Z})$ . Since  $G$  is finitely generated,  $H \xrightarrow{i} G/T$  is of finite index. Let  $H' = \pi^{-1}(H)$  where  $\pi: G \rightarrow G/T$  is the quotient homomorphism. Then  $H'$  is isomorphic to  $H \times T$  because  $i^*(e_{\mathbf{Z}}) = 0$ . In particular,  $H$  is isomorphic to a subgroup of  $G$ . Let  $\mu: H \rightarrow G$  be a monomorphism. The commutative diagram

$$\begin{array}{ccc} K_0(\mathbf{Q}H) & \xrightarrow{T_0} & HH_0(\mathbf{Q}H) \\ \mu_* \downarrow & & \mu_* \downarrow \\ K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G) \end{array}$$

and the observation that  $\mu_*(HH_0(\mathbf{Q}H))_{C(1)} \subset HH_0(\mathbf{Q}G)_{C(1)}$  and  $\mu_*(HH_0(\mathbf{Q}H)') \subset HH_0(\mathbf{Q}G)'$  imply that  $H$  has the WBP over  $\mathbf{Q}$  (and thus Property C over  $\mathbf{Q}$ ).  $\square$

Combining Propositions 5.2 and 5.3 we get:

**THEOREM 5.4.** *Suppose that  $G$  is a group of type  $\mathcal{F}$  and has the WBP over  $\mathbf{Q}$ . If  $\chi_1(G; \mathbf{Q}) \neq 0$ , then the center of  $G$  is infinite cyclic.*  $\square$

Groups of type  $\mathcal{F}$  are a very special class of torsion free groups; one would hope that all groups of type  $\mathcal{F}$  have the WBP over  $\mathbf{Q}$ . There are special classes of groups of type  $\mathcal{F}$  which are known to have the WBP over  $\mathbf{Q}$ . We recall two such classes.

A group  $G$  is a *linear group* if it is a subgroup of  $GL(n, \mathbf{K})$  where  $\mathbf{K}$  is a field of characteristic zero. Bass [Bass, Theorem 9.6] proved that a torsion free linear group has the SBP over  $\mathbf{C}$  (and thus has the WBP over  $\mathbf{Q}$ ); also see [Eck].

**COROLLARY 5.5.** *Suppose  $G$  is a linear group of type  $\mathcal{F}$ . If  $\chi_1(G; \mathbf{Q}) \neq 0$ , then the center of  $G$  is infinite cyclic.  $\square$*

Eckmann [Eck] proved that a group of cohomological dimension 2 over  $\mathbf{Q}$  has the SBP over  $\mathbf{Q}$ . Consequently:

**COROLLARY 5.6.** *Suppose  $G$  is of type  $\mathcal{F}$  and has cohomological dimension 2 over  $\mathbf{Q}$ . If  $\chi_1(G; \mathbf{Q}) \neq 0$ , then the center of  $G$  is infinite cyclic.  $\square$*

There is a sense in which we can say that  $\chi_1(G; \mathbf{Q})$  is an integer. Denote the composite homomorphism  $Z(G) \hookrightarrow G \xrightarrow{A} H_1(G; \mathbf{Z}) \rightarrow H_1(G; \mathbf{Q})$  by  $A_{\mathbf{Q}}: Z(G) \rightarrow H_1(G; \mathbf{Q})$ .

**THEOREM 5.7.** *Let  $G$  be a group of type  $\mathcal{F}$  which has the WBP over  $\mathbf{Q}$ . Then there exists an integer  $n_G$  (depending only on  $G$ ) such that  $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$ .*

*Proof.* If  $\chi_1(G; \mathbf{Q}) = 0$  take  $n_G = 0$ . If  $\chi_1(G; \mathbf{Q}) \neq 0$  then by Theorem 5.4 the center of  $G$  is infinite cyclic. Let  $\tau \in Z(G)$  generate  $Z(G)$ . Since  $\chi_1(G; \mathbf{Q}) \neq 0$  we have  $\chi_1(G; \mathbf{Q})(\tau) \neq 0$ . By Theorem 4.2,  $\chi_1(G; \mathbf{Q})(\tau) = -\chi(G/\langle \tau \rangle; \mathbf{Q})\{\tau\}$ . Then for any integer  $r$ :  $\chi_1(G; \mathbf{Q})(\tau^r) = r\chi_1(G; \mathbf{Q})(\tau) = -r\chi(G/\langle \tau \rangle; \mathbf{Q})\{\tau\} = -\chi(G/\langle \tau \rangle; \mathbf{Q})A_{\mathbf{Q}}(\tau^r)$ . Thus  $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$  with  $n_G = -\chi(G/\langle \tau \rangle; \mathbf{Q})$ .  $\square$

*Remarks.*

1. All integers occur as  $n_G$  for some  $G$ . Given  $n \in \mathbf{Z}$ , there is a group  $H$  of type  $\mathcal{F}$  with  $\chi(H) = -n$  (e.g. take  $H$  to be an appropriate Cartesian product of free groups). Let  $G = H \times T$  where  $T$  is infinite cyclic. Clearly,  $\chi(G/\langle \tau \rangle; \mathbf{Q}) = \chi(H)$  where  $\tau$  is a generator of  $(1) \times T \subset G$  and so  $\chi_1(G; \mathbf{Q}) = nA_{\mathbf{Q}}$  (alternatively, see Example 6.15).

2. Theorem 5.7 remains true without the hypothesis that  $G$  has the WBP over  $\mathbf{Q}$  although the proof is considerably more lengthy. To prove this strengthened result, one shows that for *any* group  $G$  of type  $\mathcal{F}$ :

- (a) The restriction of  $\chi_1(G; \mathbf{Q})$  to  $Z(G) \cap [G, G]$  is zero.
- (b) If  $\chi_1(G; \mathbf{Q}) \neq 0$  then  $\dim_{\mathbf{Q}} A_{\mathbf{Q}}(Z(G)) = 1$ .

The desired conclusion follows easily from (a), (b) and Theorem 4.2.

Theorem 5.7 raises the question: For what groups  $G$  of type  $\mathcal{F}$  is  $\chi_1(G, \mathbf{Q}) \neq 0$ ? We give a necessary condition. Recall that a group  $H$  has type  $\mathcal{FD}$  if there is a finitely dominated  $K(H, 1)$  (i.e.  $K(H, 1)$  is a homotopy retract of a finite complex).

**PROPOSITION 5.8.** *If  $\chi_1(G, \mathbf{Q}) \neq 0$  then  $G$  is isomorphic to a semidirect product  $\langle H, t \mid tht^{-1} = \theta(h) \text{ for all } h \in H \rangle$  where  $H$  has type  $\mathcal{FD}$ .*

*Proof.* Let  $\tau \in Z(G)$  be such that  $\chi_1(G, \mathbf{Q})(\tau) \neq 0$ . By Theorem 4.2, it follows that  $\{\tau\} \in H_1(G) \equiv G_{\text{ab}}$  is of infinite order. Thus there is an epimorphism  $p: G \rightarrow \mathbf{Z}$  with  $p(\tau) = n$  for some  $n > 0$ . Let  $H = \ker(p)$ . Since  $\tau \in Z(G)$ ,  $p^{-1}(n\mathbf{Z}) \cong H \times \mathbf{Z}$  and has finite index in  $G$ . Thus  $H \times \mathbf{Z}$  has type  $\mathcal{F}$  and so  $H$  has type  $\mathcal{FD}$ .  $\square$

Thus it is worthwhile to compute  $\chi_1(G, \mathbf{Q})$  in terms of such a semidirect product structure. The geometric problem underlying this is the study of  $\chi_1(X)$  where  $X$  is a mapping torus. We study this next, returning to the group theoretic case in §7.

## 6. MAPPING TORI

In this section, we consider  $\chi_1(X)$  and  $\tilde{\chi}_1(X)$  when  $X$  is the mapping torus of a map  $f: Z \rightarrow Z$ . The main results are Theorems 6.3, 6.13, 6.14, 6.16 and Corollary 6.18. Applications to the aspherical case will be given in §7.

Suppose  $Z$  is a path connected space and has a basepoint  $v \in Z$ . Given a continuous map  $f: Z \rightarrow Z$ , its *mapping torus*, denoted by  $T(Z, f)$ , is the space obtained from  $Z \times [0, 1]$  by identifying  $(z, 1)$  with  $(f(z), 0)$  for each  $z \in Z$ . The image of  $(z, u) \in Z \times [0, 1]$  in  $T(Z, f)$  will be denoted by  $[z, u]$ . Choose a basepath  $\sigma$  from  $v$  to  $f(v)$  and let  $\theta: H \rightarrow H$  be the self homomorphism of  $H \equiv \pi_1(Z, v)$  determined by  $f$  and  $\sigma$ .

Let  $X = T(Z, f)$ . Choose  $w = [v, 0]$  as a basepoint for  $X$  and let  $G = \pi_1(X, w)$ . There is a canonical map of  $X$  to the standard circle  $S^1$  (realized as complex numbers of unit modulus) given by:  $p_f: X \rightarrow S^1$ ,  $p_f([z, s]) = e^{2\pi i s}$ . Let  $i: Z \hookrightarrow X$  be the inclusion  $z \mapsto [z, 0]$ .