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element $H_0 \in \mathfrak{t}$. We may and shall choose the positive roots so that they take strictly positive values on H_0 . The action of W on \mathfrak{t} is generated by reflections about the kernels of the positive roots.

Since each \mathfrak{m}_i is also preserved by $ad(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{v+i}\}$ of \mathfrak{m}_i such that, for $H \in \mathfrak{t}$, the matrix of $ad(H)|_{\mathfrak{m}_i}$ with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the ad -invariance of the inner product $\langle \cdot, \cdot \rangle$ implies, for all $1 \leq i \leq v$, all $1 \leq j \leq 2v$ and all $H \in \mathfrak{t}$ that

$$\langle H, [X_i, X_j] \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+v}, X_j \rangle.$$

By orthonormality, this last pairing can only be nontrivial if $j = i + v$. Hence if $j \neq i + v$, we have $[X_i, X_j] \in \mathfrak{m}$. The same thing happens if $i > v$ and $j \neq i - v$.

On the other hand, for $1 \leq i \leq v$, set $H_i = [X_i, X_{v+i}]$. This is $Ad(T)$ -invariant, so $H_i \in \mathfrak{t}$, and $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$. It follows that the span of X_i, X_{i+v}, H_i is a Lie subalgebra \mathfrak{g}_i of \mathfrak{g} . It is always isomorphic to $\mathfrak{su}(2)$.

3. INVARIANT THEORY

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

(3.1) Let

$$\mathcal{S} = \bigoplus_{p=0}^{\infty} \mathcal{S}^p \quad \text{and} \quad \Lambda = \bigoplus_{q=0}^l \Lambda^q \quad (l = \dim \mathfrak{t})$$

be the symmetric and exterior algebras on \mathfrak{t}^* , respectively. The adjoint action of W on \mathfrak{t} induces representations of W on \mathcal{S} and Λ by degree-preserving algebra automorphisms. For example, the action of W on Λ^l is multiplication by the *sign character*

$$\varepsilon: W \rightarrow \{\pm 1\} \quad \text{given by} \quad \varepsilon(w) = \det Ad(w)_{\mathfrak{t}}.$$

Note that $\varepsilon(w)$ is the parity of the number of reflections needed to express $Ad(w)_{\mathfrak{t}}$.

We are interested in W -invariant polynomials, and more generally, W -invariant differential forms with polynomial coefficients. For the unitary group $U(n)$, the ring of invariants \mathcal{S}^W is generated by the elementary symmetric polynomials s_1, \dots, s_n in variables x_1, \dots, x_n defined as

$$s_d(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_d \leq n} x_{i_1} \cdots x_{i_d}.$$

The elementary symmetric polynomials are algebraically independent, and their number equals the dimension n of a maximal torus of $U(n)$. In general, we have

(3.2) THEOREM (Chevalley). *The ring \mathcal{S}^W has algebraically independent homogeneous generators F_1, \dots, F_l , hence is a polynomial ring*

$$\mathcal{S}^W = \mathbf{R}[F_1, \dots, F_l].$$

We number these generators so that $\deg F_1 \leq \deg F_2 \leq \dots \leq \deg F_l$. (Note to experts: Since we are not assuming G to be semisimple, some of the F_i 's could have degree one.) The *exponents* $m_1 \leq m_2 \leq \dots \leq m_l$ of W acting on \mathfrak{t} are defined by the relations $m_i + 1 = \deg F_i$. It is known that $m_1 + \dots + m_l = v$, and $(1 + m_1) \cdots (1 + m_l) = |W|$.

Every compact connected Lie group is, up to finite covering, the product of a central torus with a direct product of classical groups $SU(n)$, $SO(n)$, $Sp(n)$, and exceptional groups G_2, F_4, E_6, E_7, E_8 . For these groups the m_i 's are given as follows:

$$SU(n): 1, 2, \dots, n-1. \quad SO(2n): 1, 3, \dots, 2n-3, n-1.$$

$$SO(2n+1) \text{ and } Sp(n): 1, 3, \dots, 2n-1.$$

$$G_2: 1, 5. \quad F_4: 1, 5, 7, 11.$$

$$E_6: 1, 4, 5, 7, 8, 11.$$

$$E_7: 1, 5, 7, 9, 11, 13, 17.$$

$$E_8: 1, 7, 11, 13, 17, 19, 23, 29.$$

These numbers are easy to verify for the classical groups and G_2 (whose maximal torus T is that of $SU(3)$ with Weyl group S_3 extended by the inverse map on T), using elementary symmetric polynomials as above. Computing the exponents for the other exceptional groups is more difficult. See [C].

(3.3) The W -module structure of the whole polynomial ring \mathcal{S} is given as follows. Let \mathcal{D} be the ring of constant coefficient differential operators on \mathcal{S} . We can think of \mathcal{D} as the symmetric algebra $S(\mathfrak{t})$, where $H \in \mathfrak{t}$

corresponds to the derivation of \mathcal{S} extending the functional on \mathfrak{t}^* given by evaluation at H . Then W acts naturally on \mathcal{D} and one defines the “harmonic polynomials” in \mathcal{S} to be those annihilated by the W -invariant differential operators:

$$\mathcal{H} = \{f \in \mathcal{S} : \mathcal{D}^W f = 0\}.$$

Let $\mathcal{H}^p = \mathcal{H} \cap \mathcal{S}^p$. Then $\mathcal{H} = \bigoplus_p \mathcal{H}^p$, since a differential operator is W -invariant only if each of its homogeneous components is so. The action of W on \mathcal{S} leaves \mathcal{H} invariant.

Let \mathcal{I} be the ideal in \mathcal{S} generated by the elements of \mathcal{S}^W of positive degree. It is known (see [H, p. 360]) that $\mathcal{S} = \mathcal{H} \oplus \mathcal{I}$, and the multiplication map is a linear isomorphism $\mathcal{H} \otimes \mathcal{S}^W \xrightarrow{\sim} \mathcal{I}$. The former implies that \mathcal{S}/\mathcal{I} and \mathcal{H} are isomorphic W -modules. They are in fact isomorphic to the regular representation of W , as we shall see in (5.4). The isomorphism $\mathcal{H} \otimes \mathcal{S}^W \simeq \mathcal{I}$ implies the identity

$$\sum_{p \geq 0} \dim \mathcal{H}^p t^p = \prod_{i=1}^l (1 + t + t^2 + \cdots + t^{m_i}),$$

which in turn shows that $\dim \mathcal{H}^v = 1$, and $\mathcal{H}^p = 0$ for $p > v$.

(3.4) Let V be any irreducible W -module. Suppose V is a constituent of \mathcal{S}^b , and not a constituent of \mathcal{S}^c , for any $c < b$. We call b the *birthday* of V . Then the V -isotypic component of \mathcal{S}^b must consist of harmonic polynomials, for otherwise, a W -invariant differential operator of positive degree would intertwine V with a space of polynomials of lower degree.

For example, the *primordial* harmonic polynomial is

$$\Pi = \prod_{\alpha \in \Delta^+} \alpha \in \mathcal{H}^v,$$

where we recall that Δ^+ is the set of positive roots. For $U(n)$, Π is the van der Monde determinant $\prod_{i < j} x_i - x_j$, which transforms under the symmetric group S_n by the sign character. In general, Π transforms by the sign character ε of W , and any other polynomial transforming by ε must vanish on all root hyperplanes, hence be divisible by Π . Therefore Π is harmonic, v is the birthday of ε and (1.4) shows that \mathcal{H}^v is spanned by Π .

We say that Π is primordial because \mathcal{H} is spanned by the partial derivatives of Π (see [S]). This turns out to be the algebraic analogue of Poincaré duality for G/T .

As we have seen, the sign character is also afforded by Λ^1 . In general, if \mathfrak{g} is simple then each exterior power Λ^q is an irreducible W -module. We shall determine the birthday of each Λ^q shortly.

(3.5) Now consider the algebra $\mathcal{S} \otimes \Lambda$ of differential forms on \mathfrak{t} with polynomial coefficients. Let F_1, \dots, F_l be homogeneous generators of \mathcal{S}^W as in (3.2). Extending that result, Solomon [Sol] has described the W -invariants in $\mathcal{S} \otimes \Lambda$. Because it seems not so well known but is important here, we give a proof, taken from [H].

(3.6) THEOREM (Solomon). *The space $(\mathcal{S} \otimes \Lambda)^W$ of W -invariants in $\mathcal{S} \otimes \Lambda$ is a free \mathcal{S}^W -module with basis*

$$\{dF_{i_1} \wedge \cdots \wedge dF_{i_q} : 1 \leq i_1 < \cdots < i_q \leq l\}.$$

Proof. It is a general fact about polynomials that the algebraic independence of F_1, \dots, F_l is equivalent to the form $dF_1 \wedge \cdots \wedge dF_l$ not being identically zero. Let x_1, \dots, x_l be a basis of \mathfrak{t}^* . Then

$$dF_1 \wedge \cdots \wedge dF_l = J dx_1 \cdots dx_l,$$

where the Jacobian J is a polynomial of degree $m_1 + \cdots + m_l = v$. The left side is W -invariant and $dx_1 \wedge \cdots \wedge dx_l$ affords the sign character ε . Hence J must also afford ε and, because of its degree, J must be a nonzero multiple of the primordial harmonic polynomial Π . Thus

$$dF_1 \wedge \cdots \wedge dF_l = c \Pi dx_1 \wedge \cdots \wedge dx_l,$$

for some nonzero real number c .

For a sequence $I = i_1 < \cdots < i_q$, let I' be the increasing sequence of all integers in $\{1, \dots, l\} - \{i_1, \dots, i_q\}$. Set $dF_I = dF_{i_1} \wedge \cdots \wedge dF_{i_q}$ for any sequence I . Let k be the quotient field of \mathcal{S} . If $f_I \in k$ are such that $\sum_I f_I dF_I = 0$ then multiplying by $dF_{I'}$ kills all terms but I , leaving $\pm c f_I \Pi dx_1 \cdots dx_l = 0$, so $f_I = 0$. Counting dimensions, we find that the dF_I are a k -basis of $k \otimes \Lambda$, and are in particular linearly independent over \mathcal{S}^W . Now suppose $\omega \in \mathcal{S} \otimes \Lambda$ is W -invariant. We can express $\omega = \sum_I g_I dF_I$ for some $g_I \in k$. Multiplying by $dF_{I'}$ again, we have

$$\omega \wedge dF_{I'} = \pm c g_I \Pi dx_1 \cdots dx_l \in [\mathcal{S} \otimes \Lambda]^W.$$

This forces g_I to be not only W -invariant, but also polynomial. \square

For $\omega \in \mathcal{S} \otimes \Lambda$, let $\omega' \in \mathcal{S}/\mathcal{I} \otimes \Lambda$ be obtained by reducing the coefficients of ω modulo \mathcal{I} . This induces an exact sequence

$$0 \rightarrow (\mathcal{S} \otimes \Lambda)^W \rightarrow (\mathcal{S} \otimes \Lambda)^W \xrightarrow{\omega \mapsto \omega'} (\mathcal{S}/\mathcal{I} \otimes \Lambda)^W \rightarrow 0.$$

It follows immediately from Solomon's theorem that $\{dF'_{i_1} \wedge \cdots \wedge dF'_{i_q} : 1 \leq i_1 < \cdots < i_q \leq l\}$ spans $(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W$ (over \mathbf{R}). This is in fact a

basis, since \mathcal{S}/\mathcal{I} affords the regular representation of W , so $\dim(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W = 2^l$. We therefore have the following

(3.7) COROLLARY. $(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W$ is an exterior algebra with generators

$$dF'_i \in [(\mathcal{S}/\mathcal{I})^{m_i} \otimes \Lambda^1]^W, \quad \text{for } 1 \leq i \leq l.$$

We will see later that this exterior algebra is, with degrees in \mathcal{S}/\mathcal{I} doubled, the cohomology ring of the compact Lie group G . As W -representations, we have $\mathcal{S}/\mathcal{I} \simeq \mathcal{H}$ and the corollary gives the following

(3.8) MULTIPLICITY FORMULA.

$$\sum_{n=0}^{\infty} \dim \operatorname{Hom}_W(\Lambda^q, \mathcal{H}^n) u^n = s_q(u^{m_1}, \dots, u^{m_l}),$$

where s_q is the elementary symmetric polynomial in l -variables, and the m_i are the exponents of W .

In particular, the birthday of Λ^q is $m_1 + \dots + m_q$, if g is simple.

(3.9) We close this section with a digression. Suppose g is simple, so all Λ^q are irreducible W -modules. We can actually witness the birth of Λ^q in \mathcal{H} using the differentials dF_i , as follows. Choose a basis x_1, \dots, x_l of \mathfrak{t}^* , and consider a q -form

$$\omega = \sum f_{i_1, \dots, i_q} dx_{i_1} \wedge \dots \wedge dx_{i_q} \in \mathcal{S} \otimes \Lambda^q.$$

The linear span of the coefficient polynomials f_{i_1, \dots, i_q} is independent of the choice of basis $\{x_i\}$. Moreover, if ω is W -invariant and nonzero, then its coefficients span a W -invariant subspace of \mathcal{S} which is isomorphic to Λ^q as a W -module, since the latter is irreducible and self-contragredient.

For example, we have seen that

$$dF_1 \wedge \dots \wedge dF_l = c \Pi dx_1 \wedge \dots \wedge dx_q,$$

where c is a nonzero scalar, and Π is the primordial harmonic polynomial, affording the sign character of W . We have a generalization of this for all Λ^q .

(3.10) PROPOSITION. For $1 \leq q \leq l$, the coefficients of $dF_1 \wedge \dots \wedge dF_q$ are harmonic polynomials. They span an irreducible W -submodule of $\mathcal{H}^{m_1 + \dots + m_q}$, isomorphic to Λ^q .

Proof. The coefficients of $dF_1 \wedge \cdots \wedge dF_q \in (S^{m_1 + \cdots + m_q} \otimes \Lambda^q)^W$ span a W -invariant subspace of $S^{m_1 + \cdots + m_q}$, isomorphic to Λ^q . As in (3.4), these coefficients are harmonic because $m_1 + \cdots + m_q$ is the birthday of Λ^q , by the multiplicity formula (3.8). \square

4. INVARIANT DIFFERENTIAL FORMS

The ideas in this section go back to E. Cartan and de Rham. For a thorough exposition, see [C-E].

(4.1) Suppose a compact Lie group G acts transitively on a manifold M . Let τ_g be the diffeomorphism of M corresponding to $g \in G$. A differential p -form $\omega \in \Omega^p(M)$ is G -invariant if $\tau_g^* \omega = \omega$. Such a form is determined by its value at any one point of M . One shows by averaging that every de Rham cohomology class on M is represented by a G -invariant form, and that the subcomplex of invariant forms is preserved by the exterior derivative.

Identify $M = G/K$ where K is the stabilizer of a point $o \in M$. We have an orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$, where \mathfrak{r} is the Lie algebra of K . Moreover this decomposition is preserved by $Ad(K)$. For example if G acts on itself by left multiplication then $K = 1$ and $\mathfrak{n} = \mathfrak{g}$. For another example take $M = G/T$, so $K = T$ and $\mathfrak{n} = \mathfrak{m}$. In general, \mathfrak{n} is naturally identified with the tangent space $T_o(M)$, so an invariant form $\tilde{\omega}$ is determined by the skew-symmetric multilinear map

$$\omega = \tilde{\omega}_o: \mathfrak{n} \times \cdots \times \mathfrak{n} \rightarrow \mathbf{R}.$$

That is, $\omega \in \Lambda^p \mathfrak{n}^*$. The invariance of $\tilde{\omega}$ under K implies the $Ad(K)$ -invariance of ω . Conversely, any element $\omega \in (\Lambda^p \mathfrak{n}^*)^K$ determines a G -invariant form $\tilde{\omega}$ on M by the formula

$$\tilde{\omega}_{g \cdot o}((d\tau_g)_o X_1, \dots, (d\tau_g)_o X_p) = \omega(X_1, \dots, X_p),$$

for $X_1, \dots, X_p \in \mathfrak{n}$ and $g \in G$. Thus we may identify the G -invariant p -forms on M with the space $(\Lambda^p \mathfrak{n}^*)^K$. In this view, the exterior derivative becomes the map $\delta: (\Lambda^p \mathfrak{n}^*)^K \rightarrow (\Lambda^{p+1} \mathfrak{n}^*)^K$ given by

$$\delta \omega(X_0, \dots, X_p) = \frac{1}{p+1} \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j]_{\mathfrak{n}}, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).$$

Here $\hat{}$ means the term is omitted, and $[X_i, X_j]_{\mathfrak{n}}$ is the projection of $[X_i, X_j]$ into \mathfrak{n} along \mathfrak{r} . The complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$ computes the de Rham cohomology of M .