7. Cohomology

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

where $w_2(K)$ is a natural number which is easily computed. It follows from the results of [MW] that the odd part of (7) is true if K is abelian. This makes it possible to calculate the odd part of $\# K_2(R)$ in concrete cases: by the Kronecker-Weber theorem, K is a subfield of a cyclotomic field. From this one derives that $\zeta_K(s)$ is a product of Dirichlet series the values of which at negative integers can be expressed by generalized Bernoulli numbers. Finally, the 2-part of $\# K_2(R)$ has been calculated in some real quadratic cases by Browkin and Schinzel [BrS]. Collecting these informations, one has, e.g.,

$$\# K_2(R) = 12 \text{ for } K = \mathbf{Q}(\sqrt{6}) ,$$

([Hu3], Th. 8). Now it is not too difficult to write down sufficiently many different elements of $K_2(R)$ (so-called Steinberg and Dennis-Stein symbols). Thus, one knows $K_2(R)$, and presentations of $SL_n(R)$, $n \geqslant 3$, drop out. In [Hu2], Hurrelbrink treats the integral domains of the real subfields of the 9-th and 15-th cyclotomic field, this time relying on the Birch-Tate conjecture for these fields. A generalization of this line of thought to cases involving skew fields seems to be out of sight at present.

I would like to mention here (although K-theory is not explicitely used) a purely algebraic method due to P.M. Cohn [C] which gives presentations of $SL_2(R)$ for certain subrings R of $\mathbf{C}$; this method applies to the integral domains of the euclidean imaginary quadratic fields $\mathbf{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. The presentations involve *all* matrices

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \ y \text{ a unit} ,$$

hence are, by genesis, not finite. In the cases in question it is however possible to reduce them to finite presentations. This is carried out in [F, p. 73 ff.].

7. Cohomology

We recall some notions from the cohomology theory of groups; ideal references for our purposes are the book [Br] by K. Brown and Serre's article [Se3].

A group Γ is said to have cohomological dimension n, $cd\,\Gamma = n$, if n is the maximal dimension for which there exists a Γ-module M such that $H^n(\Gamma, M) \neq 0$. If there is no such n, $cd\,\Gamma = \infty$. If $cd\,\Gamma < \infty$, then Γ is torsion free. It is known that $cd\,\Gamma = 1$ if and only if Γ is free. There is a virtual notion: $vcd\,\Gamma = n$ if Γ contains a torsion free subgroup Δ of finite index with $cd\Delta = n$; this is independent of the choice of Δ because $cd\Delta_1 = cd\Delta_2$ for $\Delta_1 \subset \Delta_2$ torsion free with finite index.

Let $cd\Gamma = n$. Γ is called a duality group if there exists a dualizing module D such that

$$H^i(\Gamma, M) = H_{n-i}(\Gamma, D \otimes M)$$

for all i and Γ-modules M. If Γ is of type FP (a condition virtually satisfied by our unit groups) then an equivalent condition is

$$H^i(\Gamma, \mathbf{Z}\Gamma) = 0 \text{ for } i \neq n, \text{ and}$$
$$H^n(\Gamma, \mathbf{Z}\Gamma) \text{ is torsion free.}$$

The dualizing module is then $D = H^n(\Gamma, \mathbf{Z}\Gamma)$. If $D = \mathbf{Z}$, Γ is called a Poincaré duality group. The corresponding virtual notion is clear.

Now let G be a linear algebraic group, semisimple and connected, defined over $\mathbf{Q}$, and let $\Gamma \subset G$ be an arithmetic subgroup. Suppose Γ is torsion free. Let $C < G(\mathbf{R})$ be a maximal compact subgroup. Then $\Gamma \cap C = 1$. Hence Γ operates properly discontinuously on

$$X = C \backslash G(\mathbf{R}) .$$

Since X is diffeomorphic to a Euclidean space, in particular contractible, it follows that

$$X(\Gamma) := C \backslash G(\mathbf{R}) / \Gamma$$

is a $K(\Gamma, 1)$-space, that is, $\pi_1(X(\Gamma)) = \Gamma$ and $\pi_i(X(\Gamma)) = 0$ for $i > 1$. Furthermore,

$$H^*(\Gamma, -) = H^*(X(\Gamma), -) .$$

This implies that $cd\Gamma = \dim X(\Gamma)$ if $X(\Gamma)$ is compact and $< \dim X(\Gamma)$ otherwise. In the fundamental paper [BSe], Borel and Serre have shown how to enlarge X to a manifold $\bar{X}$ with "corners" on which Γ still operates properly, and for which $\bar{X}/\Gamma$ is a *compact* $K(\Gamma, 1)$-manifold with corners. The boundary is explicit enough (it has the homotopy type of a bouquet of $(l-1)$-spheres, where $l = \mathbf{Q}$-rank of G), and one derives

$$cd\Gamma = \dim X - l$$

([BSe], 11.4.3). Further, Γ is a duality group, and Poincaré if and only if $l = 0$, that is, $X(\Gamma)$ is compact.

We apply this to G = norm-1-elements of $A^\times$ and SG = elements of reduced norm 1 over the center. Then

$$\dim X(\Gamma) = r(A),$$
$$\dim X(S\Gamma) = r(SA) .$$

The $\mathbf{Q}$-rank equals $n-1$ in both cases (see [BT], 6.21. The field- and skew field part contribute nothing to the rank.) So we see from the general theory that $X(\Gamma)$ is compact if and only if $A = D$ is a skewfield. The "if" part is our Theorem 1, and for the "only if" a direct argument is available: taking $A = M_2(\mathbf{Q})$ for simplicity it is not difficult to see that the points of X represented by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \in \mathbf{Q}^\times$$

cannot be uniformly bounded by right multiplication with $SL_2(\mathbf{Z})$. Summarizing, we obtain

THEOREM 4. *Let* Γ *be a unit group. Then*

$$vcd\,\Gamma = r(A) - n + 1,$$
$$vcd\,S\Gamma = r(SA) - n + 1\,.$$

Γ *is a virtual duality group, and Poincaré if and only if* $n = 1$.

This is our second generalization of Dirichlet's unit theorem. Recall that the easy part of this theorem is

$$\Gamma/\text{torsion} \cong \mathbf{Z}^r,\ r \leqslant r(K)\,,$$

and the hard part is to show that $r = r(K)$. But from the Künneth formula one easily derives $cd\,\mathbf{Z}^r = r$. Another interesting consequence is

COROLLARY. Γ *contains a free subgroup of finite index if and only if* $A = M_2(\mathbf{Q})$.

Proof. In view of the theorem and formula (5) we have to show that the equation

$$r_1' \frac{(ns+2)\,(ns-1)}{2} + r_1'' \frac{(ns-2)\,(ns+1)}{2} + r_2(ns-1)\,(ns+1) = n$$

admits as only solution $n = 2$, $r_1' = s = 1$, $r_1'' = r_2 = 0$. First, we must have $r_2 = 0$ and next $r_1' r_1'' = 0$ because otherwise there would be two summands $> \frac{n}{2}$. The reader can work out that

$$r_1''(ns-2)\,(ns+1) = 2n$$

has no solution, the remaining equation only the one stated above.

In other words, $SL_2(\mathbf{Z})$ is not virtually isomorphic to any other unit group, and is virtually free. This latter property is usually proved by applying the Kurosh subgroup theorem to $PSL_2(\mathbf{Z}) = C_2 * C_3$.

The actual calculation of the integral cohomology $H^*(\Gamma)$ is hard. Being satisfied with virtual results, we can consider the case $\Gamma = SL_2(\mathbf{Z})$ as settled (if you don't, apply the Mayer-Vietoris sequence ([Se 3], 1.3) to $\Gamma = C_4 *_{C_2} C_6$.) The next case $\Gamma = SL_3(\mathbf{Z})$ requires already substantial work; the interested reader is referred to [So], and, for congruence groups, to [LS].

In view of Theorem 4, it is natural to ask for a classifying space for Γ of the "correct" dimension $vcd\,\Gamma$. Such a space has been constructed (for unit groups Γ) by Ash [Ash] as a deformation retract of the space of forms which was the object of reduction theory. Ash's construction, which is as elementary as ingenious, generalizes ideas pursued as early as 1907 by Voronoi; see [Br], ch. VIII for a discussion. Thereby the general but very involved construction of Borel-Serre can be avoided in the present case. Let us sketch the procedure in the simplest case $\Gamma = SL_n(\mathbf{Z})$: the space of forms is

$$H^+ = SO(n)\backslash SL_n(\mathbf{R}) ,$$

and what we eventually want, is a compact deformation retract of

$$H^+/SL_n(\mathbf{Z}) = SO(n)\backslash SL_n(\mathbf{R})/SL_n(\mathbf{Z}) .$$

Now

$$SL_n(\mathbf{R})/SL_n(\mathbf{Z}) =: G$$

is naturally identified with a space of lattices in $\mathbf{R}^n$; so instead of working with forms $\bmod SL_n(\mathbf{Z})$, we can work with lattices $\bmod SO(n)$. For $L \in G$ define

$$m(L) = \min\{\langle x, x\rangle \mid x \in L\backslash(0)\}$$

and

$$M(L) = \{x \in L \mid \langle x, x\rangle = m(L)\} ,$$

the set of "minimal vectors" of L. Ash calls L "well rounded" if $M(L)$ contains a basis of $\mathbf{R}^n$. It is clear that these definitions descend to $SO(n)\backslash G$. Ash's main result is that the space $W = \{$well-rounded lattices with $m(L) = 1\} \bmod SO(n)$ is the required deformation retract.

Returning to the Dirichlet unit theorem once more, we observe that the rank $r(K)$ is detected by the cohomology in still another way. Let us work with the full unit group and write $\Gamma = C_k \times \mathbf{Z}^r$. Using the well-known cohomology of cyclic groups and the Künneth formula, one readily computes

$$H^n(\Gamma) = H^n(\Gamma, \mathbf{Z}) = \mathbf{Z}^{\binom{r}{n}} \times (C_k)^{\binom{r}{n-2} + \binom{r}{n-4} + \cdots} ,$$

so that for $n > r$

$$H^n(\Gamma) = \begin{cases} (C_k)^{\binom{r}{0} + \binom{r}{2} + \cdots + \binom{r}{r'}}, & n \text{ even} \\ (C_k)^{\binom{r}{1} + \binom{r}{3} + \cdots + \binom{r}{r''}}, & n \text{ odd}, \end{cases}$$

where $r' = r$ for r even and $r' = r - 1$ for r odd, and vice versa for r''. Thus r is recovered from $H^*(\Gamma)$ in two ways:

(1) $r = \max\{n \mid rkH^n(\Gamma) > 0\}$,

(2) $r + 1 =$ lowest dimension from which on $H^*(\Gamma)$ is periodic (of period two).

The periodicity has been generalized by Venkov [Ve] to orders in skew fields. He proves the following general theorem: Let G be a connected noncompact Lie group and $\Gamma < G$ a discrete subgroup with the properties

(i) *every finite subgroup of* Γ *has cohomological period* g;

(ii) *there is* $c \in H^g(\Gamma)$ *such that, for every finite subgroup* $H < \Gamma$, $res^{\Gamma}_{H}c$ *generates* $H^g(H)$.

Then the cup product by c *gives isomorphisms*

$$H^k(\Gamma, M) \cong H^{k+g}(\Gamma, M)$$

for all Γ*-modules* M *and* $k > \dim G - \dim C$, C *a maximal compact subgroup.*

This too can be applied to $G =$ norm-1-group of $A^\times$, where $A = D$ has to be a skew field. G is noncompact unless D is a totally definite quaternion algebra, a case which we can happily omit from our considerations because in this case $S\Gamma$ is finite. The possible finite subgroups of $D^\times$ have been classified by Amitsur [Am]. As his results show, their Sylow groups are cyclic or generalized quaternion groups; hence they have periodic cohomology ([Br], th. VI 9.5). Since Γ contains — up to isomorphism — only finitely many finite subgroups, these have a common period. These arguments are not even necessary because Venkov shows ([Ve] Prop. 5) that, for $g = \dim_Q D$,

$$H^g(H) = \mathbf{Z}/|H|\mathbf{Z}$$

for any finite subgroup H of Γ; this implies g-periodicity by ([Br], Th. VI 9.1). The harder part is condition (ii) which requires a spectral sequence argument. One obtains

THEOREM 5. *Suppose that* $A = D$ *is not a totally definite quaternion algebra. Then there are isomorphisms*

$$H^k(\Gamma, M) \cong H^{k+g}(\Gamma, M)$$

for all Γ*-modules and* M *and* $k > r(D)$. *Moreover,* $r(D)+1$ *is the smallest dimension from which on* $H^*(\Gamma, -)$ *is n-periodic.*

The last statement follows from the facts that (1) Γ is a virtual duality group and hence $H^{r(D)}(\Gamma, \mathbf{Z}\Gamma)$ is torsion free ([Br], VIII 11.2) whereas (2) for any group Γ with $vcd\,\Gamma = k$, $H^m(\Gamma)$ is torsion for $m > k$ ([Se3], p. 101).

It should be possible to refine the period g in the theorem by one more directly derived from the periods of the finite subgroups of Γ as in the number field case where the period equals 2 if there are nontrivial torsion units of norm 1 (in which case n must be even!).

REMARK. We have seen (in the general case, Γ torsionfree) that

$$H^*(\Gamma, -) = H^*(X(\Gamma), -)\ .$$

Taking real coefficients (with trivial Γ-action) the latter groups are, by de Rham's theorem, given by differential forms on $X(\Gamma)$; these in turn correspond to Γ-automorphic forms on X. In this way, the real cohomology of Γ becomes part of the theory of automorphic forms.

8. CONGRUENCE SUBGROUPS AND NORMAL SUBGROUPS

Recall that we have defined

$$\Gamma(m) = \text{kernel of } (\Gamma \to (\Lambda/m\Lambda)^\times)\ ,$$

the congruence subgroup of level m of Γ. Obviously $\Gamma(m)$ has finite index in Γ. The following question is classical: does every subgroup of finite index of Γ contain a congruence group?

Let us say that Γ satisfies (CP) if this is so. Let $\Lambda \subset \Lambda'$. If Γ' satisfies (CP), so does Γ. To prove the converse, it suffices to show that every $\Gamma(n)$ contains a $\Gamma'(m)$. This will be so if Γ contains a $\Gamma'(m)$. But there is $m \in \mathbf{N}$ with $m\Lambda' \subset \Lambda$, and this implies $\Gamma'(m) \subset \Lambda \cap \Gamma' = \Gamma$. Thus, property (CP) depends only on A.

For $A = K$ a number field, (CP) has essentially been proved by Chevalley [Ch]. Let $H < R^\times$ be of finite index, and $H_0 < R^\times$ any congruence subgroup. Then $H_0^k \subset H$ for some $k \in \mathbf{N}$; so it suffices to show