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UNITS OF CLASSICAL ORDERS: A SURVEY

by Ernst KLEINERT

ABSTRACT: This survey describes the principal methods and results in the theory of units of orders.

CONTENTS

1. INTRODUCTION

We consider units of orders in semisimple algebras A of finite dimension over $\mathbf{Q}$. The "algebraic background" for this (a formulation due to Zassenhaus) is the classical theory of algebras, where we find as basic results the Wedderburn decomposition plus the exact sequence

$$1 \to B(K) \to \prod_{\mathfrak{p}} B(K_{\mathfrak{p}}) \to \mathbf{Q}/\mathbf{Z} \to 0 \, ,$$

K denoting a number field and $B(K)$ the Brauer group. (For notation and commentary, see [R], §32). This sequence, which has been called the "Main Theorem in the theory of algebras" ([N], p. 244), in fact contains a full classification of the algebras under consideration and is surely one of the most substantial abbreviations of pure mathematics, incorporating the Hasse norm theorem (exactness on the left for cyclic algebras) as well as the reciprocity law for the norm residue symbol (exactness in the middle) and, implicitly, Hasse's classification of local skew fields (exactness on the right). By an *order* $\Lambda \subset A$ we mean a subring containing 1, consisting of $\mathbf{Z}$-integral elements and such that $\mathbf{Q}\Lambda = A$. Natural examples of orders are the integral domains $\mathcal{O}_K$ of number fields K, full matrix rings over $\mathcal{O}_K$, crossed product orders $(\mathcal{O}_L/\mathcal{O}_K, f)$, where L/K is Galois with group G and $f \in H^2(G, \mathcal{O}_L^\times)$ a factor system with values in $\mathcal{O}_L^\times$, and, as a non-simple example, the integral group ring $\mathbf{Z}G$, G finite. The arithmetic of such Λ has two natural parts, namely the theories of modules and of units. The module theory, known as integral representation theory, has been developed systematically and has grown out powerful techniques; lattices over orders enter in various class groups which in turn figure in canonical sequences, or live in almost split sequences which can be arranged to Auslander-Reiten quivers. For maximal orders, the lattice theory can be reduced to a ray class group in the central field, by theorems of Eichler and Swan, and is thus passed to algebraic number theory. In the general case, however, we are at least able to tell why the subject is hopeless: most orders have wildly infinite representation type, which means that their lattices cannot be classified by presently existing methods.

The unit theory is not in such a state, and we still have to subscribe to Eichler's statement in the introduction to his 1935 paper [E1]: "Allein die Einheitentheorie ist noch in keiner Weise abgerundet." There are still very few general results which substantially add to the basic information that unit groups of orders are finitely presentable. This can, of course, not be ascribed to a lack of interest. The point is that classically the interest in integral matrix groups has been concentrated on reduction theory, quotient manifolds and automorphic forms rather than on "the groups themselves." It seems characteristic that neither Siegel in his definite paper [S1] (where he "finished the job", as Weyl put it) nor Weyl in his paraphrase [W] explicitly mention the finite generation, let alone the finite presentation which is effectively proved there; their interest was in finiteness theorems concerning reduction theory of quadratic forms. These classical lines of research have, as everyone knows, been pursued further and have led to the vast and deep generalizations now established in the theory of arithmetic groups, an immense body of methods and results and a meeting ground for more or less the whole apparatus of number theory, algebraic geometry, topology and analysis. Yet it looks strange that for many years no paper seems to have appeared combining the words "units" and "orders" in the title. And it is noteworthy in this connection that most of the results (significant exception: the Bass unit theorem) generalize to larger classes of arithmetic groups and thereby ignore the fact that $\Lambda^\times$ is the unit group of a ring — surely a strong condition on a group. For instance, it should be fruitful to study the natural map $\mathbf{Z}\Lambda^\times \to \Lambda$ from the integral group ring.

The purpose of this survey is to collect the principal methods and theorems about units of orders as far as they refer "more directly" to the structure of these groups (I am aware of the fact that this phrase is not well defined). Scattered and incomplete as the results may be they surely deserve to be presented in some sort of connection. Let us view Dirichlet's unit theorem as a starting point; we will describe three generalizations of it (Theorems 1, 4, 9) which arise from its topological, cohomological, and K-theoretical aspects. In the last section, we present some thoughts about what should be expected from a "General Unit Theorem" which would have satisfied Eichler — certainly a long range project. The reader will also come across a number of more concrete problems which can be attacked with reasonable hope of success.

Any reader will miss something in a survey on a theme which stretches over more or less the whole area of pure mathematics. (I will be grateful to receive criticism as well as hints to further results which fit the theme). On the other side, there will be few to whom I can offer absolutely no news. I readily admit that there are more competent mathematicians who could have written a survey on this subject; however, *non possunt omnes omnia.*

The following notation will be used throughout (unless otherwise specified): A is a semisimple algebra of finite dimension over $\mathbf{Q}$. If A is simple, we write $A = M_n(D)$, D the skewfield part, $K = Z(A) = Z(D)$ the center and $R = \mathcal{O}_K$ the ring of integers of K. $\Lambda \subset A$ is a $\mathbf{Z}$-order (equivalently, R-order) in A, $\Gamma = \Lambda^\times$ the unit group. We exclude from our considerations S-arithmetical and local cases (the former causing complications, the latter being wholly different). Also, we do not treat the specific problems and results for integral group rings which come from the existence of a group base and require special techniques — the isomorphism problem and the Zassenhaus conjecture in its various forms, which are a world of their own and for which I refer to [Ro], and the results due to Hoechsmann, Ritter, Sehgal, and others concerning generators of subgroups of finite index in $(\mathbf{Z}G)^\times$, for which I refer to Sehgal's book [Seh].

I would like to thank Jean-Pierre Serre for his comments on an earlier version of this paper.

2. Elementary Properties

(1) Let N denote the regular norm $A^\times \to \mathbf{Q}^\times$. It is easy to see that $\Gamma = \{x \in \Lambda \mid N^2(x) = 1\}$.

If we specify a $\mathbf{Z}$-basis of Λ, this becomes a polynomial equation in the coefficients of x with respect to this basis, and the elements of Γ correspond precisely to the integral solutions. This shows that Γ is an *arithmetic group*, and thereby makes available all the general results on this class of groups. (A reference ideally suited to the present theme is Serre's survey article [Se4]; we also mention [Pl].) In fact, a good deal of the present paper will be concerned with specifying the general results to the case of unit groups.

(2) *Let* $\Lambda \subset \Lambda'$ *be orders in* A *with unit groups* Γ, Γ'. *Then* $\Gamma = \Gamma' \cap \Lambda$, *and* $|\Gamma' : \Gamma|$ *is finite.*

Proof. For any $x \in \Gamma$ we have $x^{-1} \in \mathbf{Z}[x]$ since x is a zero of a monic integral polynomial with constant term ± 1. This proves the first statement. For the second, assume that, for $x, y \in \Gamma'$, we have

$$x - y = mz, \quad \text{where} \quad m = |\Lambda' : \Lambda|, z \in \Lambda' .$$

Then

$$xy^{-1} = 1 + mzy^{-1} \in \Gamma' \cap \Lambda = \Gamma .$$

This shows that

$$|\Gamma' : \Gamma| \leqslant |\Lambda' : m\Lambda| = (\dim A)^m .$$

(2) allows us to reduce all questions concerning virtual properties of Γ to arbitrary orders in simple algebras. (A group is said to have a property virtually if a subgroup of finite index has that property). Finite presentability is such a property: if $\Gamma_0 \subset \Gamma$, $|\Gamma : \Gamma_0|$ finite, has a finite presentation, then so has, by Reidemeister-Schreier, the intersection of its conjugates, which is normal; now use the fact that the class of finitely presented groups is closed under extensions ([J], p. 187, Th. 1).

(3) Γ *is virtually torsion free.*

Proof. It is easy to see that there is an upper bound, and consequently a lowest common multiple N for the orders of torsion elements $x \in A^\times$; all such $x \neq 1$ satisfy

$$x^{N-1} + x^{N-2} + \dots + x + 1 = 0 .$$

For $n \in \mathbf{N}$ let

$$\Gamma(n) = \text{kernel of } (\Gamma \to (\Lambda/n\Lambda)^{\times})$$

the congruence group mod n; this is a normal subgroup of finite index. Obviously $\Gamma(n)$ is torsion free for $n > N$. With more effort, one can do much better: the regular representation injects $\Gamma(n)$ into the congruence group mod n in $GL_m(\mathbf{Z})$, $m = \dim A$, and Minkowski has shown that this is torsion free for $n > 2$ [Mi].

(4) Γ *contains only finitely many isomorphism classes of finite subgroups.*

Proof. If $\Gamma_0 < \Gamma$ is torsion free and normal of finite index, then every finite subgroup of Γ is isomorphic to a subgroup of Γ/Γ_0.

Later, we will show more: Γ contains only finitely many conjugacy classes of finite subgroups.

(5) Γ *is residually finite, that is, for every* $x \in \Gamma, x \neq 1$, *there is a normal subgroup* Γ_0 *of finite index such that* $x \notin \Gamma_0$.

Of course, almost all $\Gamma(n)$ will do. It follows that Γ is hopfian, that is, not isomorphic to a proper factor group (see [MKS], p. 116).

(6) Finally, let us mention here the following result due to Zassenhaus [Z2] (although it is not entirely elementary): Γ contains a solvable subgroup of finite index if and only if the Wedderburn components of A are number fields or definite quaternions over $\mathbf{Q}$.

Sketch of proof: the problem is readily reduced to simple A. The "If" part is then trivial.

Conversely, if matrices are involved, one knows that Γ has infinitely many subfactor groups of the form $SL_n(F)$, where F is a finite field. The same is therefore true of any subgroup of finite index. In the skew field case, the argument is more intricate; we refer to [Z2].

3. Finite generation: classical reduction theory

The most basic fact about Γ is that it is finitely generated; this is even valid for arbitrary arithmetic groups, as has been proved by A. Borel and Harish-Chandra in the fundamental paper [BHC]. Here I shall describe the classical approach, carried out by Siegel [S1], who completed earlier work of

Minkowski, Humbert, Weyl and Eichler. The leading idea is to make Γ operate on a suitable topological space; if this operation is "good enough", then generators can be read off form it, even, as we shall see in the next section, defining relations. Let us begin with the basic definitions.

Let the group H operate on the topological space T as a group of homeomorphisms. For a non-empty subset $F \subset T$ define

$$E(F) = \{h \in H \mid F \cap Fh \neq \varnothing\} .$$

If we think of F as a fundamental domain, then $E(F)$ consists of those elements which carry F to a "neighbor". The following basic observation occurs in [S1, section 9].

BASIC LEMMA. *Assume that*

(i) $FH = T$;

(ii) $FE(F)$ *is a neighborhood of* F; *and*

(iii) T *is connected.*

Then $E = E(F)$ *generates* H.

Proof. Let H_0 be the subgroup generated by E and $\{h_i\}$ be a set of right coset representatives of H mod H_0. Then the sets $X_i = FH_0 h_i$ are disjoint, open and form a cover of T. Since T is connected, there can be only one of them.

Let us illustrate this at once with the most classical case of $H = \Gamma = SL_n(\mathbf{Z})$. In accordance with previous terminology, this is half the unit group. In order to obtain finite generation one has to find T and F such that F is not too small (otherwise (i) or (ii) might fail) and not too large (otherwise E might be infinite).

A plausible condition for E being finite is that H operates discontinuously, that is, no H-orbit has a cluster point. (If x is a cluster point of fH, write $x = f'h, f' \in F$; if there is a neighborhood $f' \in U \subset F$, then Uh contains infinitely many fh_i and $h_i h^{-1} \in E$). This rules out the most near-at-hand choice of T, the natural space $\mathbf{R}^n$. (Convince yourself for $n = 2$, that Γ does not operate discontinuously on $\mathbf{R}^2$!) A possible choice, however, is $T = G = SL_n(\mathbf{R})$, Γ operating by right multiplication. Γ is a discrete subgroup of G. For $t > 0$, $w > 0$ define

$$\begin{aligned} D_t &= \{\operatorname{diag}(a_1, \ldots, a_n) \in G \mid 0 < a_i \leqslant t a_{i+1}\} \\ N_w &= \{(u_{ij}) \text{ strict upper triangular with } |n_{ij}| \leqslant w\} \end{aligned}$$

and

$$S_{t,w} = SO(n) D_t N_w \subset G .$$

This is called a "Siegel domain". One now proves two things:

(1) for $t \geqslant 2/\sqrt{3}$, $w \geqslant \frac{1}{2}$ we have $S_{t,w}\Gamma = G$;

(2) for all t, w the set

$$\{\gamma \in \Gamma \mid S_{t,w} \cap S_{t,w}\gamma \neq \varnothing\}$$

is finite.

Proofs can be found, e.g., in Borel's book [B2, § 1]. It is not difficult to apply the lemma, and hence Γ is finitely generated.

Remark on terminology: by a fundamental domain we mean a set containing a system of orbit representatives and such that H-translates of it intersect at most on the boundaries. It is equivalent to (i) of the Basic lemma that F contains a fundamental domain. Property (2) (and its generalizations) is called "Siegel's property" by Borel; (1) and (2) constitute what Borel calls "ensemble fondamental". Other authors require other properties or distinguish between "fundamental set" and "fundamental region". Note that a Siegel domain is not a fundamental domain in this sense! See [Te, 4.4] for Minkowski's classical fundamental domain of $SL_n(\mathbf{Z})$.

Let us briefly indicate (although this goes beyond our theme) how the argument generalizes to arithmetic groups. $SO(n)$ is a maximal compact subgroup of G, the set D of diagonal matrices in G is a maximal torus (a torus is a group isomorphic to a direct product of copies of $\mathbf{R}^\times$), and the set N of strict upper triangular matrices is a maximal unipotent subgroup of G. Such groups are reasonably unique, and one has the Iwasawa decomposition

$$\begin{array}{lcl} SO(n) \times D \times N & \tilde{\to} & G \\ (o, d, n) & \to & odn, \end{array}$$

which is a diffeomorphism of manifolds. Let D operate by conjugation on the vector space $\mathfrak{g}$ consisting of n-by-n matrices of trace zero, the Lie algebra of G. The character group $\operatorname{Hom}(D, \mathbf{R}^\times)$ is generated, say, by the first $n-1$ coordinate functions and is isomorphic to $\mathbf{Z}^{n-1}$; for a character λ define

$$\mathfrak{g}^\lambda = \{x \in \mathfrak{g} \mid dxd^{-1} = \lambda(d)x, \quad \text{all} \quad d \in D\}$$

and call λ a root if $\mathfrak{g}^\lambda \neq 0$. Among the roots one can distinguish simple roots which can be choosen to be

$$\lambda_i\colon \operatorname{diag}(d_1, \ldots, d_n) \to d_i d_{i+1}^{-1}.$$

Thus,

$$D_t = \{d \in D \mid \lambda(d) \leqslant t, \lambda \text{ simple}, d \text{ positive}\}.$$

N_w is simply a "generic" compact subset of N. Now all of these concepts — maximal compact subgroups, tori, unipotent subgroups, Iwasawa decomposition, roots and simple roots — generalize to reductive real algebraic groups G. Hence Siegel domains can be defined completely analogously, and one can prove the analogues of (1) and (2) for arithmetic subgroups Γ of G; this has been done in [BHC]. By elementary property (1), this applies to unit groups of orders.

Secondly, let us pursue the connection of these concepts with reduction of quadratic forms. In applying the lemma it is natural to look for a manifold of least possible dimension which possesses a suitable F. In the case of $\Gamma = SL_n(\mathbf{Z})$, the observation that $SO(n) \cap \Gamma$ = compact and discrete, hence finite leads to the expectation that the operation of Γ on the coset space $SO(n)\backslash G$ still does the job. By linear algebra, the map

$$\pi : \begin{cases} G \to \text{symmetric positive matrices of determinant 1} \\ g \to g^t g \end{cases}$$

is surjective; this implies that the operation of G on these matrices, $(g, x) \to g^t x g$, is transitive. The stabilizer of 1_n is $SO(n)$; hence $SO(n)\backslash G$ identifies with that space, which in turn is identified with the space of positive definite quadratic forms of determinant 1. If $g = kdn$ is the Iwasawa decomposition, then from

$$\pi(g) = n^t dk^t kdn = n^t d^2 n$$

we see that $S_{t,w}$ is mapped to the Siegel domain

$$S^t_{t^2,w} = \{n^t dn \mid d \in D_{t^2},\ n \in N_w\}$$

in the space of forms. Hence (1) translates to Minkowski's "reduction theorem" saying that every positive form of determinant 1 is a Γ-translate of an element of $S'_{4/3,1/2}$. It is clear that $E(S'_{t^2,w})$ is still finite.

Hence we can (in principle) obtain a finite set of generators from the Γ-operation on a space of dimension

$$n^2 - 1 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1 .$$

But now the attentive reader will object that this is somewhat like putting the cart before the horse because reduction theory doubtless has an interest in its own right whereas it is elementary to write down a finite set of generators for $SL_n(\mathbf{Z})$. In fact, such a set can be given for $SL_n(R)$ if R is euclidean and finite over $\mathbf{Z}$ (see [Ne], p. 107) and for $SL_n(\mathbf{Z})$ one can do with

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & 0 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ (-1)^{n-1} & & & & & 0 \end{pmatrix},$$

as Hua and Reiner have shown [HR]. Hurwitz [H] treated $SL_2(R)$, where R is the integral domain of a number field, and remarked that the procedure can be generalized to $SL_n(R)$, giving a sketch for $n = 3$. The most general form of the argument was given by O'Meara [O'M]. The finite generation of $SL_n(R)$ can be derived directly from the finiteness conditions incorporated in the notion of number field, and there is no need to employ the geometry. This should also hold for the case in which skew fields are involved although a purely algebraic treatment of this case has — as far as I know — not been given.

The reply is that finite generation as such is a very weak information and gives hardly any insight into the structure of our unit groups. It is the *raison d'être* of groups to operate on sets having an internal structure, and it is by understanding the operation that we understand groups. With regard to units of orders, this has been stressed by Eichler [E1]:

„Von der Überzeugung ausgehend, daß die Begriffswelt der Zahlengeometrie die geeignete Grundlage für den Aufbau eines tragenden Gerüsts für die hyperkomplexe Einheitentheorie abgibt, beschäftige ich mich hier mit Darstellungen der Einheitengruppen durch affine Abbildungen eines Raumes auf sich. In dieser geometrischen Gestalt trat sie erstmals in der analytischen Zahlentheorie auf und führte auf geometrische Untersuchungen, die bis heute nicht in befriedigender Weise abgeschlossen werden konnten. Die Hauptaufgabe der Einheitentheorie sehe ich nun in der Auffindung von Invarianten dieser Abbildungsgruppen."

Needless to say, this is still the adequate view on units of orders. Furthermore, as we shall see later, the geometric method leads at least theoretically to defining relations among the generators thus found; in the only case where these can be derived purely algebraically ($SL_2(\mathbf{Z})$) this derivation has an artificial and a-posteriori character, and doubtless the most natural way to the presentation

$$SL_2(\mathbf{Z}) = C_4 *_{C_2} C_6$$

is by letting the group operate on a tree [Se1].

Moving towards general orders we first deal with the case where $A = D$ is a skewfield, in which the number geometric method works particularly smoothly. Put $D_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} D$ and

$$G = \{x \in D_{\mathbf{R}}^{\times} \mid N(x)^2 = 1\} ,$$

N denoting the regular norm $D \to \mathbf{Q}$. Clearly, $\Gamma \subset G$ is a discrete subgroup. The following result was proved by Käte Hey in her doctoral thesis (Hamburg 1929) and reappears in [Sch], [E1], and [Z1].

THEOREM 1. *G/Γ is compact.*

Proof (according to [Z1]). We work with a $\mathbf{Z}$-basis of Λ, so that in $D_{\mathbf{R}} = \mathbf{R}^g$, $g = \dim D$, Λ appears as $\mathbf{Z}^g$.

Let C be any convex, 0-symmetric compact set in $\mathbf{R}^g$ such that $\mathrm{vol}(C) > 2^g$. By Minkowski's lattice point theorem, C contains a nonzero $a \in \Lambda$. If $x \in G$, then $\mathrm{vol}(Cx) = \mathrm{vol}(C)$ because of $|N(x)| = 1$, and Cx is still convex and 0-symmetric, hence contains a nonzero $a_x \in \Lambda$.

Now let (x_n) be a sequence of elements in G. Then there are $a_j \in \Lambda \backslash \{0\}$ such that

$$a_i = c_i x_i , \ c_i \in C .$$

It follows that $|N(a_i)|$ is bounded because N is bounded on C. Because D is a skew field, we have

$$|N(a_i)| = |\Lambda : a_i \Lambda| \neq 0 .$$

Since there are only finitely many right ideals of bounded index, there is a subsequence (a_k) such that

$$a_k \Lambda = a_1 \Lambda \text{ (say)} ,$$

hence

$$a_k = a_1 \varepsilon_k , \ \varepsilon_k \in \Gamma .$$

Further,

$$|N(c_k)| = |N(a_k)| = |N(a_1)| > 0 .$$

Since C is compact, (c_k) contains a convergent subsequence (c_l). The last inequality shows that (c_l^{-1}) is convergent. From

$$x_l \varepsilon_l^{-1} = c_l^{-1} a_1$$

we now read off that G/Γ is compact. Note that we have used, so to speak, only half of the lattice point theorem in that there was no need to specify C.

This is our first generalization of Dirichlet's unit theorem, the most classical result on units of orders, in that it contains what one calls the hard part of this theorem. In fact, let $D = K$ be a number field and write, in usual notations,

$$K_{\mathbf{R}} = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}, \; r_1 + 2r_2 = g \; ;$$

we have

$$G = \{(x_1, \ldots, x_{r_1+r_2}) \in K_{\mathbf{R}} \mid (x_1 \ldots x_{r_1}) \mid x_{r_1+1} \mid^2 \ldots \mid x_{r_1+r_2} \mid^2 = 1\} .$$

The logarithm map

$$\log \begin{cases} G & \to \mathbf{R}^r, r = r_1 + r_2 - 1 \\ (x_i) & \to (\log |x_1|, \ldots, \log |x_{r_1}|, 2\log |x_{r_1+1}|, \ldots, 2\log |x_{r_1+r_2-1}|) \end{cases}$$

is a homomorphism, continuous, surjective and has compact kernel. Since Γ is discrete in G, $\log | \Gamma$ has finite kernel, and $\log \Gamma$ is discrete in $\log G = \mathbf{R}^r$, hence a lattice. It follows that

$$\Gamma \cong W(K) \times \mathbf{Z}^{\tilde{r}}, \; \tilde{r} = rk \log \Gamma \leqslant r ,$$

$W(K)$ denoting the roots of unity in K. This is the easy part of Dirichlet's theorem, the hard one being that $\tilde{r} = r$. In the standard presentations of the theorem, one now has to go through some unperspicous trickery (involving, of course, the lattice point theorem) in order to establish the existence of sufficiently many independent units. But clearly $\tilde{r} = r$ is equivalent to the compactness of $\log G/\log \Gamma$, which follows at once from Theorem 1.

The generalization of Theorem 1 to arithmetic groups is as follows: let $G \subset Gl_n$ be a reductive algebraic group defined over $\mathbf{Q}$, Γ an arithmetic subgroup. Then $G_{\mathbf{R}}/\Gamma$ is compact if and only if G^0 (= connected component of identity) has no nontrivial $\mathbf{Q}$-characters and all elements of $G_{\mathbf{Q}}$ are semisimple (see [B2], p. 55 ff.). The reader might try to verify that the hypotheses of this result are satisfied if G is the algebraic group defined over $\mathbf{Q}$ by the norm-1-elements of a skew field.

The finite presentation of Γ can be extracted from Theorem 1. Let $K \subset G$ be a maximal compact subgroup; then $\Gamma \cap K$ is finite, hence Γ contains a subgroup Γ_0 of finite index such that $\Gamma_0 \cap K = 1$. Then $K \backslash G / \Gamma_0$ is a compact manifold, and since $K \backslash G$ is a homeomorphic to a Euclidean space, Γ_0 is its fundamental group. But the fundamental group of a compact manifold is always finitely presented (a proof of this fact can be found in [Ra], p. 95).

The two "extreme cases" $A = M_n(\mathbf{Q})$ and $A = D$ are comparatively easy; unfortunately, the general case offers difficulties which cannot be overcome by a straightforward combination of these two. (Be sure to see clearly why the skew field property of D is indispensable in the proof of Theorem 1). However, Zassenhaus proves the following generalization in which A is not even required to be semisimple: there is a system F of right coset representatives of G mod Γ of the following form:

$$F = \{xW(x)VW(x)^{-1}\},$$

where the x run over a compact subset of G, $W(x) \in G$ is a function with finite range and V a torus with positive diagonal elements. Visibly, there is a resemblance to a Siegel domain. In the skewfield case, $V = 1$. From this one can derive the finite presentability of Γ along classical lines (see section 4).

Approaching the general case, now we could simply refer the reader to Borel's text [B2] since there is no point in reporting at length on the contents of a textbook which is standard since 25 years. On the other hand, even in a survey article the reader will expect to become acquainted more closely with the methods. Therefore let us consider in some detail Siegel's classical treatment. Actually, we follow Weyl [W] who found it necessary to provide a careful explanation of Siegel's "all too laconic" arguments. He divided the proof (of finite generation) into three "theorems of finiteness"; we will lead the discussion up to a point where the content and the rôle of these theorems become visible. Perhaps the clarity and elegance of Weyl's arguments is still of more than merely historical interest.

Let $A = M_n(D)$. A *lattice* N in D^n is a finitely generated $\mathbf{Z}$-module containing a D-basis of the right D-vector space D^n. Such a basis, $\mathscr{D} = \{d_1, \ldots, d_n\}$, is called a *semibasis* of N. Given $\mathscr{D}$, the set

$$L(\mathscr{D}) = \{(a_1, \ldots, a_n)^t \in D^n \mid \Sigma d_i a_i \in N\}$$

is another lattice, containing the standard basis vectors $e_1, \ldots, e_n$. $L(\mathscr{D})$ is called the representation of N in terms of $\mathscr{D}$, and all such $L(\mathscr{D})$ are called *admissible* lattices. The left order

$$O_l(N, A) = \{x \in A \mid xN \subseteq N\}$$

is our order Λ, and

$$\Gamma = \{x \in A \mid xN = N\} = \Lambda^\times$$

is the group which interests us; Weyl calls it the *lattice group*. If $\mathscr{D}$, $\mathscr{D}'$ are two semibases, then $L(\mathscr{D}) = L(\mathscr{D}')$ if and only if $\mathscr{D}' = s\mathscr{D}'$ for some $s \in \Gamma$.

An $\mathbf{R}$-basis of $D_\mathbf{R} = \mathbf{R} \otimes_\mathbf{Q} D$ is called *normal* if the regular representation R of $D_\mathbf{R}$ with respect to that basis has the property

$$R(D_\mathbf{R}) = R(D_\mathbf{R})^t \quad (t \text{ denoting transpose}).$$

It is not difficult to establish the existence of normal bases: let $K = Z(D)$ and write as before

$$\mathbf{R} \otimes_{\mathbf{Q}} K = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}, \ r_1 + 2r_2 = \dim_Q K \,.$$

Then

$$D_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} D = (\mathbf{R} \otimes_{\mathbf{Q}} K) \otimes_K D = \prod_{i=1}^{r_1} \mathbf{R} \otimes_K^i D \times \prod_{j=1}^{r_2} \mathbf{C} \otimes_K^j D \,,$$

where $\otimes^i(\otimes^j)$ indicates that the tensor product is to be formed with respect to the K-module structure of $\mathbf{R}(\mathbf{C})$ corresponding to the i-th (j-th) embedding of K into $\mathbf{R}(\mathbf{C})$. The $\mathbf{C} \otimes^j D$ are central simple $\mathbf{C}$-algebras and hence full matrix rings over $\mathbf{C}$. The $\mathbf{R} \otimes^i D$ are central simple $\mathbf{R}$-algebras and hence full matrix rings over $\mathbf{R}$ or $\mathbf{H}$, the quaternion skew field. More precisely, if $s^2 = \dim_K D$,

$$(3) \qquad \begin{cases} \mathbf{C} \otimes^j D \cong M_s(\mathbf{C}) & \\ \mathbf{R} \otimes^i D \cong M_s(\mathbf{R}), & \text{for } i = 1, \dots, r_1' \text{ (say)} \\ \mathbf{R} \otimes^i D \cong M_{s/2}(\mathbf{H}), & \text{for } i = r_1' + 1, \dots, r_1' + r_1'' = r_1 \,. \end{cases}$$

If we now replace the elements of $\mathbf{C}$ and $\mathbf{H}$ by their regular representations with respect to the standard bases, then the typical elements are

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbf{C}, \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \in \mathbf{H} \,,$$

and transposing corresponds to the usual conjugation on $\mathbf{C}$ and $\mathbf{H}$. Combining this with the fact that for any skew field F, the regular representation of $M_n(F)$ over F is equivalent to n times the identity, we see that normal bases exist.

We fix one of them and obtain a conjugation on $D_{\mathbf{R}}$ by

$$\alpha \to \bar{\alpha} = R^{-1}(R(\alpha)^t) \,.$$

Call α symmetric if $\alpha = \bar{\alpha}$, positive if $R(\alpha) > 0$ is positive definite. A *quadratic form* over $D_{\mathbf{R}}$ is now a matrix

$$F = (\gamma_{ij}) \in M_n(D_{\mathbf{R}}), \quad \text{with } \overline{\gamma_{ji}} = \gamma_{ij} \,.$$

For $x \in (D_{\mathbf{R}})^n$ put

$$F[x] = \bar{x}^t F x \,,$$

a symmetric element of $D_{\mathbf{R}}$. F is called *positive* if the real matrix $(R(\gamma_{ij}))$ is positive definite. Important note: it would not work to define $F > 0$ by $F[x] > 0$, all $x \neq 0$. Choosing $x = (0, \ldots, x_i, \ldots, 0)$, we had to have $\bar{x}_i \gamma_{ii} x_i > 0$, all $x_i \neq 0$ which implies

$$N(x_i)^2 N(\gamma_{ii}) > 0 \; ;$$

but if $D_{\mathbf{R}}$ is not a skew field, there will be $x_i \neq 0$ with $N(x_i) = 0$. The positive F form an open convex cone in the space of all forms, in particular a manifold of the same dimension. We call it H^+. Weyl shows next that $F > 0$ if and only if $F = \bar{A}^t A, A \in Gl_n(D_{\mathbf{R}})$; this implies that, as long as the conjugation $\alpha \to \bar{\alpha}$ is fixed, positiveness does not depend on the choice of a normal basis.

Let $Tr: D \to \mathbf{Q}$ (or $D_{\mathbf{R}} \to \mathbf{R}$) denote the trace of the regular representation. It is not hard to show that, if F is positive in the above sense, one has

$$t_F(x) := TrF[x] > 0, \quad \text{for } x \neq 0 \text{ in } (D_{\mathbf{R}})^n \; .$$

This is the correct definition of "positive form over a skew field"; as Weyl points out, a crucial step in Siegel's proof.

So far we have been setting the stage; now we come to the first main step, the method of successive minima originally invented by Minkowski. Let the lattice N and the positive form $t = t_F$ be given. Since for any real $s > 0$ there are only finitely many $d \in N$ with $t[d] < s$, t takes a minimum on N, say $t[d_1] = s_1$. Inductively, we define a semibasis $\mathscr{D} = \{d_1, \ldots, d_n\}$ of N by the requirement

$$t[d_m] = \min\{t[d] \mid d \in N \backslash [d_1, \ldots, d_{m-1}]\} \; ,$$

where $[d, \ldots, d_{m-1}]$ denotes the D-span of $d_1, \ldots, d_{m-1}$. Write $t[d_m] = s_m$; then $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$. We say that $\mathscr{D}$ is *reduced with respect to t.* Now we make the change of variables which transforms d_i to the unit vector e_i and N to $L(\mathscr{D})$; the new form is again denoted t. We then have

$$t[x] \geqslant t[e_m] = s_m$$

for $x \in L(\mathscr{D}) \backslash [e_1, \ldots, e_{m-1}]$ that is, $(x_m, \ldots, x_n) \neq 0$. An arbitrary form satisfying these inequalities is called $L(\mathscr{D})$*-reduced.* We have now reached a point where we can state the finiteness theorems.

I. *There exist L-reduced forms for only finitely many admissible lattices L.*

In other words: if we fix N, but run over all positive F, only finitely many lattices $L(\mathscr{D})$ are produced by the method of successive minima.

If L is admissible, call

$$Z(L) := \{F \in H^+ \mid t_F \text{ is } L - \text{reduced}\}$$

the *cell* of L. If $Z(L)$ is not empty, it is defined by infinitely many inequalities.

II. $Z(L)$ *can actually be defined by finitely many of them.*

The proof also shows that different cells have disjoint sets of inner points.

We now come to what Weyl calls the "pattern of cells"; it is only here that our lattice group Γ comes into play. Every semibasis $\mathscr{D} = \{d_1, \ldots, d_n\}$ of N determines a cell $Z(\mathscr{D})$ of reduced forms:

$$F \in Z(\mathscr{D}) \Leftrightarrow t_F[x] \geqslant t_F[d_m] \quad \text{for all } x \in N \backslash [d_1, \ldots, d_{m-1}] .$$

If we associate with $Z(\mathscr{D})$ the admissible lattice $L(Z) = L(\mathscr{D})$, then

$$\begin{aligned} L(Z) = L(Z') \Leftrightarrow L(\mathscr{D}) &= L(\mathscr{D}') \\ \Leftrightarrow \mathscr{D}' &= s\mathscr{D}, \text{ some } s \in \Gamma \\ \Leftrightarrow Z' &= sZ , \end{aligned}$$

where Γ operates on the forms in the usual manner:

$$st(x) = t(s^{-1}x), \; x \in (D_{\mathbf{R}})^n .$$

Fix once and for all finitely many semibases $\mathscr{D}_1, \ldots, \mathscr{D}_r$ such that $L(\mathscr{D}_1), \ldots, L(\mathscr{D}_r)$ are all the admissible lattices having reduced forms. If F is any form, F determines a semibasis $\mathscr{D}$ such that $L(\mathscr{D})$ has a reduced form. Hence there is $s \in \Gamma$ and i such that $\mathscr{D} = s\mathscr{D}_i$ and $sF \in Z(\mathscr{D}_i)$. In other words, the union

$$Z_0 = \bigcup_i Z(\mathscr{D}_i)$$

is a fundamental domain for the operation of Γ on the space H^+. The "Third Theorem of Finiteness", or the "Theorem of Discontinuity", shows that Z_0 has only finitely many neighbors. More precisely, Weyl defines, for any given semibasis $\mathscr{D}$ and real numbers $p \geqslant 1$, $w > 0$, a subset $H(\mathscr{D}, p, w)$ of H^+ with the following properties:

(i) for $p > 1$, $w > 0$, $H(\mathscr{D}, p, w)$ contains an open neighborhood of $Z(\mathscr{D})$;

(ii) if $p > p'$, $w > w'$, then

$$H(\mathscr{D}, p, w) \supset H(\mathscr{D}, p', w'), \quad \text{and} \quad H^+ = \bigcup_{p,w} H(\mathscr{D}, p, w) .$$

III. *Given any cell* Z, $\mathscr{D}, p$ *and* w, *the set*

$$\{s \in \Gamma \mid sZ \cap H(\mathscr{D}, p, w) \neq \varnothing\}$$

is finite.

The latter clearly implies that

$$E(Z_0) = \{s \in \Gamma \mid sZ_0 \cap Z_0 \neq \varnothing\}$$

is finite. Let us check condition (ii) of the basic lemma. There is a union $\tilde{H}$ of finitely many $H(\mathscr{D}, p, w)$ containing an open neighborhood U of Z_0. Then there are only finitely many $s \in \Gamma$ with $sZ_0 \cap \tilde{H} \neq \varnothing$. If all of these are in $E(Z_0)$, $U \subset E(Z_0)Z_0$ because every point of U is a Γ-translate of a point of Z_0. Let $s_1, \ldots, s_r$ be those not in $E(Z_0)$. Since $s_i Z_0$ and Z_0 are disjoint, closed, and H^+ is a normal space, there is an open $U_i \supset Z_0$ with $U_i \cap s_i Z_0 = \varnothing$. Then we can take $\bigcap_i U_i$.

To sum up: for the operation of Γ on H^+ there is a closed connected fundamental domain with finitely many neighbors, satisfying condition (ii) of the basic lemma. The finite generation of Γ is thereby proved; in the next section we will also extract finite presentability from the reduction theory.

We now turn to the question of minimal dimension mentioned earlier. Our space H^+ is the image of $GL_n(D_{\mathbf{R}})$ under the map $A \to \bar{A}^t A$. According to (3),

$$GL_n(D_{\mathbf{R}}) \cong GL_{ns}(\mathbf{R})^{r_1'} \times GL_{ns/2}(\mathbf{H})^{r_1''} \times GL_{ns}(\mathbf{C})^{r_2},$$

and H^+ arises by dividing out the product of the orthogonal, symplectic, and unitary groups, respectively, which are maximal compact. For $\mathbf{K} \in \{\mathbf{R}, \mathbf{H}, \mathbf{C}\}$, the real dimensions of the maximal compact subgroup of $GL_m(\mathbf{K})$ are

$$\frac{m(m-1)}{2}, \quad m(2m+1) \text{ and } m^2 .$$

A simple calculation now shows that

$$\text{(4)} \qquad \begin{aligned} \dim H^+ &= r_1' \frac{k(k+1)}{2} + r_1'' \frac{k(k-1)}{2} + r_2 k^2 \\ &=: r(A) + 1 \end{aligned}$$

where $k = ns$. In view of $N\Gamma \subset \{\pm 1\}$, the number $r(A)$ may be called the *geometric unit rank* of A; of course, for $A = K$, that is, $k = 1$, it coincides with the unit rank $r(K) = r_1 + r_2 - 1$ in the sense of number theory. Siegel shows that $r(A)$ is in fact the minimal dimension for a discontinuous action of Γ in a sense which we now explain.

Let more generally G be a locally compact topological group with a countable basis for the topology, $H < G$ a discrete subgroup and ν a Haar measure. Suppose that F is a set of coset representatives of G/H such that (a) F is a Borel set, and (b) $\nu(F) < \infty$. Siegel's first main result is

THEOREM. *In this situation,* H *operates discontinuously on the homogeneous space* $C\backslash G$ *if and only if* C *is a compact subgroup of* G.

First we have to check the hypotheses. By what has been said about the cells, (a) is easy; (b) by no means. We only sketch the proof in the case of $SL_n(\mathbf{Z})$ (see [B2], 1.11). Of course, it suffices to show that the Siegel domain

$$S_{t,w} = SO(n) \cdot D_t \cdot N_w$$

has finite volume in the Haar measure. Transferring the Haar measure to the factors of the Iwasawa decomposition, this comes down to the finiteness of

$$\int_{D_t} p(a)\,da\,,$$

where da is the Haar measure on the torus and

$$p((a_i)) = \prod_{i<j} a_i/a_j\,;$$

and this is not hard.

REMARKS

(1) The general finiteness criterion for the fundamental domain of arithmetic groups is that the underlying algebraic group has no $\mathbf{Q}$-characters ([Bo2], 12.5); that is, "half" the compactness criterion.

(2) It seems that the exact value of the volume has not yet been calculated in the general case although Weyl ([W], p. 263) hints at the possibility. It is of course known for $SL_n(\mathbf{Z})$ and some other cases; we refer to [Te, 4.4.4].

The theorem now shows that Γ cannot operate discontinuously on homogeneous spaces of $GL_n(D_{\mathbf{R}})$ of smaller dimension; a result stated already by Eichler [E2]. Of course, this does not rule out Γ-operations on spaces of smaller dimension which do not extend to the surrounding Lie group. In fact, such operations may be viewed as the basis of the cohomological results to which we come later.

The following simplification, however, is near at hand. Let R be the integral domain of the central field K and $S\Gamma$ be kernel of the reduced norm map $Nr\colon A^\times \to K^\times$, restricted to Γ (we will recall the definition of Nr in section 9). Then $(R^{\times ns}) = NrR^\times \subset Nr\Gamma$, and one deduces that $S\Gamma \cdot R^\times$, an almost direct product, has finite index in Γ. Since we don't care about finite indices and consider $R^\times$ as known by Dirichlet's theorem, we may concentrate on $S\Gamma$. In our previous notation (3), $S\Gamma$ is a discrete subgroup of

$$\prod^{r_1'} SL_{ns}(\mathbf{R}) \times \prod^{r_1''} SL_{ns/2}(\mathbf{H}) \times \prod^{r_2} SL_{ns}(\mathbf{C})$$

(where for $\mathbf{H}$, SL denotes elements of GL of reduced norm 1). Dividing out the maximal compact subgroups, we find that $S\Gamma$ operates discontinuously on a homogeneous space of dimension

$$r(SA) := r(A) - r(K) ,$$

which may be called the "*reduced geometric unit rank of A*". Explicitly, inferring

$$r(K) = r_1' + r_1'' + r_2 - 1 ,$$

we obtain from (4) the formula

$$(5) \qquad r(SA) = r_1' \frac{(k+2)(k-1)}{2} + r_1'' \frac{(k-2)(k+1)}{2} + r_2(k-1)(k+1) .$$

We will go through the cases of small $r(SA)$ in the concluding section.

It is surprising how easily the existence of a fundamental domain with finitely many neighbors implies another finiteness theorem, which has already been mentioned:

THEOREM 2. Γ *contains only finitely many conjugacy classes of finite subgroups.*

Proof [B1]. Let $G = Gl_n(D_{\mathbf{R}})$ and C be the maximal compact subgroup used above. Let $H < \Gamma$ be a finite subgroup. Then H is contained in a maximal compact $\tilde{C}$, which is conjugate to $C : \tilde{C} = gCg^{-1}$. Then $Cg^{-1}\tilde{C} = Cg^{-1}$, so H fixes the point $P = Cg^{-1}$ of $C \backslash G = H^+$. Let $\gamma \in \Gamma$ be such that $P\gamma \in Z_0$, the fundamental domain. It follows that $P\gamma\gamma^{-1}H\gamma = P\gamma$, so $\gamma^{-1}H\gamma \subset E(Z_0)$, which is finite. (This proof holds for arbitrary arithmetic groups.)

4. Presentations I: The theory of transformation groups

We have already indicated that not only generators but also defining relations can be extracted from a "good" operation of Γ on a "good" space and that reduction theory provides us with both. The basic idea is already inherent in Poincaré's treatment of Fuchsian groups (see e.g. [F], p. 168 ff.). Gerstenhaber [G] established the abstract setting; later contributions are due to Behr [Be 1, Be 2] and Macbeath [Mb]. Abels [A] gave a unified and generalized treatment; the following is taken from there.

Let $T, H, F, E = E(F)$ be as in the beginning of the last section. Let $\tilde{H}$ be the abstract group generated by elements t_h, $h \in E$, with defining relations

$$t_{h_2} \cdot t_{h_1}^{-1} = t_{h_2 h_1^{-1}} = \text{ for all } h_1, h_2 \in E$$
$$\text{such that } Fh_1 \cap Fh_2 \cap F \neq \emptyset .$$

(Be sure that this makes sense!) There is an obvious homomorphism

$$\varphi : \tilde{H} \to H,\ t_h \to h ,$$

which is surjective if the hypotheses of the basic lemma are fulfilled. With a little more luck, φ is an isomorphism.

THEOREM (Gerstenhaber – Behr – Macbeath). *Assume that F and T are connected and T is simply connected. Then φ is an isomorphism if*

(1) $\mathring{F}H = T$ ($\mathring{F}$ = interior of F)

or if

(2) $FH = T$, F is closed and $\{Fh \mid h \in H\}$ is a locally finite cover of T

or if

(3) $\{Fh \mid h \in H\}$ is an H-denumerable cover of T.

(For the definition of "H-denumerable cover" and some examples, see [A].) In particular, if one of the three cases is given and E is finite, then H is finitely presented.

The idea of the proof is the following. On the space

$$Z = \{(t, h) \in T \times \tilde{H} \mid th^{-1} \in F\}$$

($\tilde{H}$ operating on T via φ) define an equivalence relation by

$$(t_1, h_1) \sim (t_2, h_2) \Leftrightarrow t_1 = t_2 \quad \text{and} \quad \varphi(h_2 h_1^{-1}) \in E .$$

Then $Y = Z/\sim$ turns out to be a covering space of T with the properties

(i) $\tilde{H}$ acts on Y, and p: class of $(t, h) \to t$ is a $\tilde{H}$-map;

(ii) ker φ acts as a group of decktransformations of p; this action is free and transitive on the fibres of p;

(iii) There is a section for p over F.

This easily implies that φ is injective if Y is the trivial covering of T, and this will be so if T is simply connected. If T is not simply connected, one can say, under suitable hypotheses, something about the kernel of φ; Swan [Sw] constructs an exact sequence

$$1 \to N \to \pi_1(T) \to \tilde{H} \to H \to 1$$

where the kernel N can be described.

It is case (2) of the theorem which directly applies to our unit groups Γ. The space of positive forms on which we made Γ operate is an open convex cone in a Euclidean space and, as such, clearly connected and simply connected. It was pointed out that the fundamental domain could be chosen as the connected union of closed cells. The "third theorem of finiteness" ensures that $F\Gamma$ is a locally finite cover. Thus, we finally have

THEOREM 3. *The group of units of any* $\mathbf{Z}$*-order is finitely presented.*

As mentioned before, this generalizes to arithmetic groups [BHC].

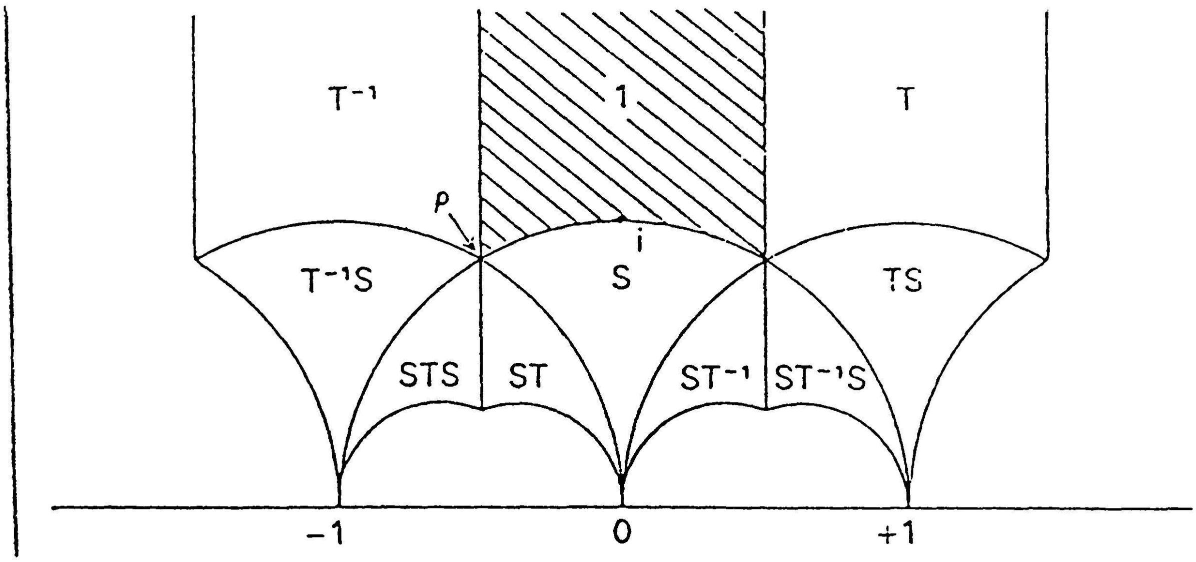
Let us illustrate the principle with the most classical example $\Gamma = SL_2(\mathbf{Z})$ mod. (± 1), $T = \mathscr{H} =$ upper half plane, $F =$ closure of the well-known fundamental domain,

$$F = \left\{ z \in \mathscr{H} \,\middle|\, |z| \geqslant 1, \, -\frac{1}{2} \leqslant Re\, z \leqslant \frac{1}{2} \right\}.$$

(This fits into our general approach since $\mathscr{H}$ is identified with $SO_2(\mathbf{R}) \backslash SL_2(\mathbf{R})$.) Evidently the prerequisites for the theorem are given. Put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod (\pm 1), \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bmod (\pm 1).$$

We get the well-known picture



of the fundamental domain and its neighbours (taken from [Se2], p. 78). Thus $\tilde{H}$ has 9 generators and a relation $t_{A^{-1}}t_B = t_{A^{-1}B}$ for every pair A, B in the set $\{T, TS, \ldots, T^{-1}S, T^{-1}\}$ such that $FA \cap FB \cap F \neq 0$. This happens only if FA, FB, F meet at ρ or ρT, and we get $2\left[\binom{5}{2}\right] = 20$ relations. It is a puzzle (elementary, but tedious) to derive the presentation

$$\tilde{H} \cong \Gamma/(\pm 1) = \langle S, ST \mid S^2 = (ST)^3 = 1 \rangle = C_2 * C_3 .$$

Admittedly this wouldn't be too easy if the result were not known in advance; but the method works in principle.

To derive presentations along these classical lines is a hard piece of work and has been done only in "small" cases. Swan [Sw] considers $SL_2(R)$, where R is the integral domain of an imaginary quadratic field K. $PSL_2(R)$ is called a Bianchi group, after L. Bianchi who was the first to embark on a systematic study of these groups 100 years ago. (See his *Opere,* Vol. 1, ed. by Bompiani and Sansone, Rome 1952; in particular Sansone's introduction to this part of Bianchi's work on page 185 ff. The fundamental domain below appears on p. 239. This particular case had already been treated by Picard.) Here the geometric unit rank $r(A) = r(SA)$ is 3, the dimension of the space of positive Hermitian forms over $\mathbf{C}$ with discriminant 1, or hyperbolic 3-space. Swan obtains presentations for $K = \mathbf{Q}(\sqrt{-d})$, with $d = 1, 2, 3, 5, 6, 11, 15, 19$. For the sake of illustration, here is the first case: define

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $SL_2(\mathbf{Z}[i])$ is generated by these matrices, and defining relations are

$$TU = UT,\ J^2 = 1,\ J \text{ central},\ L^2 = (TL)^2 = (UL)^2 = (AL)^2 = A^2$$
$$= (TA)^3 = (UAL)^3 = J .$$

If one identifies the hyperbolic 3-space with $\mathbf{R}^3$, a fundamental domain is

$$F = \left\{ (x, y, z) \,\middle|\, x \in \left[-\frac{1}{2}, \frac{1}{2}\right), \ y \in \left[0, \frac{1}{2}\right), \ z > 0, x^2 + y^2 + z^2 \geqslant 1 \right\}.$$

It is interesting to note that the section $y = 0$ of F equals the classical fundamental domain of $SL_2(\mathbf{Z})$ in the upper half plane.

Developing Swan's techniques, Frohman and Fine were able to establish a structure theorem for $SL_2(R)$ which we now describe (following [F]). Let $K = \mathbf{Q}(\sqrt{-d})$ with $d > 0$ a squarefree integer. We exclude here

$d = 1, 2, 3, 7, 11$; these give the euclidean R and require special treatment. The main theorem (6.3.1) is a decomposition as a free product with amalgamation

$$SL_2(R) = PE_2 *_F G(R) .$$

Here (4.8.2),

PE_2 = image in $PSL_2(R)$ of the subgroup generated by elementary matrices
$\cong (\mathbf{Z} \times \mathbf{Z}) *_{\mathbf{Z}} PSL_2(\mathbf{Z})$;

amazingly, this group is independent of d. Likewise, F is explicitly presented (6.3.4) and independent of d. The precise structure of $G(R)$ is not yet fully clear. The monograph [F] contains many more results, e.g. on finite subgroups and normal subgroups. A fact worth mentioning: $SL_2(R)$ contains non-congruence subgroups of finite index ([Se5]).

Some examples of the "analogous" but deeply different case in which R is the integral domain of a *real* quadratic field were treated by Kirchheimer and Wolfart [KW]; these groups are known as Hilbert modular groups. Here, $r(A) = 5$, but $r(SA) = 4$, and in fact $PSL_2(R)$ operates most naturally on a product of two upper halfplanes.

A fundamental domain has been described already by Siegel [S2]. The problems with the boundary become already for small discriminants so considerable as to require the use of machine calculations. The main result of [KW] is as follows: let ε be the fundamental unit of R and put

$$T(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \ a \in R, \ E = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Then one always has the relations

$$\begin{aligned} T(a)T(b) &= T(b)T(a), a, b \in R, \\ ET(a)E^{-1} &= T(\varepsilon^2 a), a \in R, \\ S^2 = (ES)^2 &= (ST(1))^3 = (EST(\varepsilon))^3 = 1 . \end{aligned}$$

It is shown: if $K = \mathbf{Q}(\sqrt{d})$, $d = 5, 12, 13$, then these relations are defining relations. (For $d = 3$, one additional relation is required.) In [Ki] Kirchheimer treats $SL_2(R)$ for arbitrary totally real R of class number 1; the example $R = \mathbf{Z}[\xi + \xi^{-1}]$, $\xi = e^{2\pi i/7}$ is presented in detail.

5. PRESENTATIONS II: INDEFINITE QUATERNIONS OVER THE RATIONALS

Suppose that H operates discontinuously on the manifold T. If the operation is in addition fixed-point free, then every $t \in T$ has an open neighbourhood U such that $U \cap Uh = \emptyset$ for $h \neq 1$, and one says that H operates properly discontinuously. The orbit space $X = T/H$ is then a manifold, and if T is simply connected, H is the fundamental group of X. If X belongs to a class of manifolds the fundamental groups of which are known from other sources, then we know H. Using this principle, Eichler [E 1] obtained a description of the unit groups of orders in indefinite quaternion skew fields D over $\mathbf{Q}$. (In the definite case, the unit groups are finite.)

We begin by recalling a few facts from the arithmetic of such D. Let Λ be a maximal order in D. We want to make sure that Γ contains no torsion elements except ± 1. This will be the case if D does not contain the 4-th and 6-th roots of unity (the only ones of degree 2 over $\mathbf{Q}$). For this, it is sufficient that discr Λ contains a prime factor $\equiv 1$ mod 4 and one $\equiv 1$ mod 3. Namely, let $K = \mathbf{Q}(i)$. Then $K \subset D$ if and only if K splits D. If p is a prime ramified in D (that is, dividing discr Λ), then K splits D at p if and only if $|\mathbf{Q}_p(i) : \mathbf{Q}_p| = 2$, and this is equivalent to $p \equiv 3$ mod 4. For the field of 6-th roots of unity, one argues analogously. So we make the above assumption. The only element of order 2 in the norm-one-group $S\Gamma$ is -1 (because if there were another one, it would generate a subfield containing two elements of order 2), and $PS\Gamma = S\Gamma$ mod (± 1) is torsion free.

By assumption, $D_{\mathbf{R}} \cong M_2(\mathbf{R})$, and the isomorphism maps $S\Gamma$ to a discrete subgroup of $SL_2(\mathbf{R})$. $PS\Gamma$ operates discontinously, and in the well-known manner, on the space $H^+ = SO(2)\backslash SL_2(\mathbf{R})$, which is identified with the upper half-plane. The operation is fixed-point free, because the stabilizer of a point would be in the intersection $SO(2) \cap S\Gamma = (\pm 1)$. Hence $X = H^+/PS\Gamma$ is an oriented surface. By Theorem 1, X is compact. The compact oriented surfaces and their fundamental groups are well-known; we have a presentation

$$PS\Gamma = \pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \Pi[a_i, b_i] = 1 \rangle .$$

It remains to determine the genus g, which, as the cognoscenti will guess, turns out to be a function of the discriminant. This is accomplished by Eichler (following Hey) with a truly marvellous argument, which we now describe.

Let F_0 be a fundamental domain of $S\Gamma$ in $SL_2(\mathbf{R})$. The cone $C(F_0)$ is then a fundamental domain of $S\Gamma$ in $M_2(\mathbf{R}) = D_{\mathbf{R}}$. Let

$$F = \{x \in C(F_0) \mid -1 \leqslant nr(x) \leqslant 0\} .$$

The idea is to calculate vol F (in Lebesgue measure) in two ways. The first way is to show that vol F is the residue at $s = 1$ of the zeta function of D. This rest (a) on the fact that Λ is a principal ideal domain (see e.g. [R], 35.6), and (b) on a theorem of Dirichlet, which expresses the residue at $S = 1$ of certain functions of "zeta type", associated to a lattice in Euclidean space, by the determinant of the lattice; see [BS], p. 344. Since the zeta function is known (see e.g. [De], p. 130), one gets

$$\operatorname{vol} F = \frac{\pi^2}{12} \frac{\varphi(d)}{d} .$$

(A general formula has been obtained by Käte Hey; cf. the discussion in [De], p. 133.) Here d denotes the fundamental number of D, i.e. the product of the ramified primes, which equals the square root of $|\operatorname{discr} \Lambda|$.

For the second calculation, view D as a cyclic crossed product

$$D = (L \mid \mathbf{Q}, \text{ complex conjugation}) ,$$

$L/\mathbf{Q}$ imaginary quadratic. Then one can write

$$D_{\mathbf{R}} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbf{C} \right\} ,$$

and in this representation $S\Gamma$ operates on the unit circle in $\mathbf{C}$. In the integral for vol F, two of the integrations can be carried out, and there remains an integral over a fundamental domain for $S\Gamma$ in the unit circle, with respect to the invariant measure. But for this, one has the Gauss-Bonnet formula. The final result is

$$g = \frac{\varphi(d)}{12} + 1 .$$

If $S\Gamma$ contain nontrivial torsion elements, one may apply a variant of this reasoning to a torsion-free congruence subgroup.

Soon afterwards, Hull [Hul] gave another treatment, avoiding the analytic argument but making fuller use of the theory of Fuchsian groups; this has the advantage that torsion elements cause no additional problems. The core of the arguments is the genus formula

$$2 - 2g = \upsilon + \frac{1}{2} e_2 + \frac{2}{3} e_3 ,$$

where υ is the volume of a fundamental polygon in the upper half plane, and e_i denote the number of elliptic cycles of angles $2\pi/i$. For υ, there is a formula due to Humbert. The e_i correspond to conjugacy classes of elements of order i in $PS\Gamma$, these in turn to classes of embeddings of fourth and sixth roots of unity into D; there are formulae for these as well. For an updated presentation of all of this, we refer to [Vi].

Meanwhile, Eichler's somewhat breathtaking «tour de force» has been turned into a standard argument with the calculation of a Tamagawa number as its core. Here is a rough sketch. Denote by G the algebraic group (linear, semisimple, anisotropic) defined over $\mathbf{Z}$ by the norm one elements of $D^\times$; thus, $G(\mathbf{Z}) = S\Gamma$ and $G(\mathbf{R}) = SL_2(\mathbf{R})$. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$ and view $G(\mathbf{Q})$ and $G(\mathbf{Z})$ as subgroups of $G(\mathbf{A})$ via the diagonal embedding. Let

$$C = \prod_{p \text{ prime}} G(\mathbf{Z}_p) \quad \text{and} \quad U = G(\mathbf{R}) \times C\,.$$

Then

$$G(\mathbf{A}) = G(\mathbf{Q})U \quad \text{and} \quad G(\mathbf{Q}) \cap U = G(\mathbf{Z})\,.$$

This induces a bijection of homogeneous spaces

$$G(\mathbf{A})/G(\mathbf{Q}) \cong U/G(\mathbf{Z})\,,$$

preserving the volumes with respect to the Tamagawa measure. Now the volume on the left is, by definition, the Tamagawa number, and equals 1, whence the equation

$$\operatorname{vol}(G(\mathbf{R})/G(\mathbf{Z})) = (\operatorname{vol} C)^{-1}\,.$$

Here, the volume on the right is easy and equals $\zeta(2)\varphi(d)d^{-1}$. The left side can be translated into the volume of a fundamental of $G(\mathbf{Z})$ in the upper half plane, and Gauss-Bonnet brings in the genus. The details can be found in [Vi, ch. IV].

6. Presentations III: K_2

As a byproduct of their computations, Kirchheimer and Wolfart [KW] obtained a description of $K_2(R)$ for the rings R they treated. Conversely, if $K_2(R)$ happens to be known from another source, one can derive presentations of $SL_n(R)$, $n \geqslant 3$. This idea has been pursued in a series of papers by Hurrelbrink ([Hu 1]-[Hu 3]). The general argument runs as follows.

Let R be any commutative ring, $n \geqslant 3$, and for $r \in R$ let $e_{ij}(r)$ be the elementary matrix in $SL_n(R)$ having r in the $i-j$-position ($i \neq j$). Then we have the "trivial" relations

$$(6) \qquad \begin{cases} e_{ij}(s)e_{ij}(r) & = \ e_{ij}(s+r), \\ [e_{ij}(s), e_{jl}(r)] & = \ e_{il}(sr), i \neq l \\ [e_{ij}(s), e_{kl}(r)] & = \ 1, j \neq k, i \neq l\,. \end{cases}$$

Let $St_n(R)$ be the abstract group generated by elements $x_{ij}(s)$, $s \in R$, with relations as in (6). $St_n(R)$ is called the n-th Steinberg group, and there is an obvious surjective homomorphism

$$\varphi_n = St_n(R) \to E_n(R)\ ,$$

$E_n(R)$ denoting the subgroup of $SL_n(R)$ generated by the $e_{ij}(r)$. The kernel of φ_n is denoted $K_2(n, R)$. As for GL_2 we can form the direct limit

$$St(R) = \varinjlim St_n(R)$$

and obtain a surjection

$$\varphi = \lim \varphi_n : St(R) \to E(R) = \lim E_n(R)$$

with kernel

$$K_2(R) = \lim K_2(n, R)\ .$$

Thus $K_2(R)$ codifies the nontrivial relations among elementary matrices over R of *all* sizes. Now let R be the integral domain of a number field. Here we have two stability results: Vaserstein [Va] showed that

$$E_n(R) = SL_n(R), \qquad \text{for } n \geqslant 3\ ,$$

and van der Kallen [Ka] that

$$K_2(n, r) = K_2(R), \qquad \text{for } n \geqslant 3\ ,$$

both under the hypothesis that $R^\times$ is infinite, thus excluding $R = \mathbf{Z}$ and the imaginary quadratic case.

Consequently, if one knows generators of $K_2(R)$ in terms of the $x_{ij}(s)$, one can write down immediately presentations of $SL_n(R)$, $n \geqslant 3$. Now how can one possibly know something about $K_2(R)$ without knowing the matrix relations in advance? The miracle happens in form of the Birch-Tate conjecture: assume that $K = \operatorname{Quot} R$ is totally real. Let $\zeta_K(s)$ be the Dedekind zeta function of K. The Birch-Tate conjecture predicts that

$$(7) \qquad \# K_2(R) = w_2(K)\,|\,\zeta_K(-1)\,|\,,$$

where $w_2(K)$ is a natural number which is easily computed. It follows from the results of [MW] that the odd part of (7) is true if K is abelian. This makes it possible to calculate the odd part of $\# K_2(R)$ in concrete cases: by the Kronecker-Weber theorem, K is a subfield of a cyclotomic field. From this one derives that $\zeta_K(s)$ is a product of Dirichlet series the values of which at negative integers can be expressed by generalized Bernoulli numbers. Finally, the 2-part of $\# K_2(R)$ has been calculated in some real quadratic cases by Browkin and Schinzel [BrS]. Collecting these informations, one has, e.g.,

$$\# K_2(R) = 12 \text{ for } K = \mathbf{Q}(\sqrt{6}) ,$$

([Hu 3], Th. 8). Now it is not too difficult to write down sufficiently many different elements of $K_2(R)$ (so-called Steinberg and Dennis-Stein symbols). Thus, one knows $K_2(R)$, and presentations of $SL_n(R)$, $n \geqslant 3$, drop out. In [Hu 2], Hurrelbrink treats the integral domains of the real subfields of the 9-th and 15-th cyclotomic field, this time relying on the Birch-Tate conjecture for these fields. A generalization of this line of thought to cases involving skew fields seems to be out of sight at present.

I would like to mention here (although K-theory is not explicitely used) a purely algebraic method due to P.M. Cohn [C] which gives presentations of $SL_2(R)$ for certain subrings R of $\mathbf{C}$; this method applies to the integral domains of the euclidean imaginary quadratic fields $\mathbf{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. The presentations involve *all* matrices

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \ y \text{ a unit} ,$$

hence are, by genesis, not finite. In the cases in question it is however possible to reduce them to finite presentations. This is carried out in [F, p. 73 ff.].

7. Cohomology

We recall some notions from the cohomology theory of groups; ideal references for our purposes are the book [Br] by K. Brown and Serre's article [Se 3].

A group Γ is said to have cohomological dimension n, $cd\,\Gamma = n$, if n is the maximal dimension for which there exists a Γ-module M such that $H^n(\Gamma, M) \neq 0$. If there is no such n, $cd\,\Gamma = \infty$. If $cd\,\Gamma < \infty$, then Γ is torsion free. It is known that $cd\,\Gamma = 1$ if and only if Γ is free. There is a virtual notion: $vcd\,\Gamma = n$ if Γ contains a torsion free subgroup Δ of finite index with $cd\Delta = n$; this is independent of the choice of Δ because $cd\Delta_1 = cd\Delta_2$ for $\Delta_1 \subset \Delta_2$ torsion free with finite index.

Let $cd\Gamma = n$. Γ is called a duality group if there exists a dualizing module D such that

$$H^i(\Gamma, M) = H_{n-i}(\Gamma, D \otimes M)$$

for all i and Γ-modules M. If Γ is of type FP (a condition virtually satisfied by our unit groups) then an equivalent condition is

$$\begin{aligned} &H^i(\Gamma, \mathbf{Z}\Gamma) = 0 \text{ for } i \neq n, \text{ and} \\ &H^n(\Gamma, \mathbf{Z}\Gamma) \text{ is torsion free.} \end{aligned}$$

The dualizing module is then $D = H^n(\Gamma, \mathbf{Z}\Gamma)$. If $D = \mathbf{Z}$, Γ is called a Poincaré duality group. The corresponding virtual notion is clear.

Now let G be a linear algebraic group, semisimple and connected, defined over $\mathbf{Q}$, and let $\Gamma \subset G$ be an arithmetic subgroup. Suppose Γ is torsion free. Let $C < G(\mathbf{R})$ be a maximal compact subgroup. Then $\Gamma \cap C = 1$. Hence Γ operates properly discontinuously on

$$X = C \backslash G(\mathbf{R}) .$$

Since X is diffeomorphic to a Euclidean space, in particular contractible, it follows that

$$X(\Gamma) := C \backslash G(\mathbf{R}) / \Gamma$$

is a $K(\Gamma, 1)$-space, that is, $\pi_1(X(\Gamma)) = \Gamma$ and $\pi_i(X(\Gamma)) = 0$ for $i > 1$. Furthermore,

$$H^*(\Gamma, -) = H^*(X(\Gamma), -) .$$

This implies that $cd\Gamma = \dim X(\Gamma)$ if $X(\Gamma)$ is compact and $< \dim X(\Gamma)$ otherwise. In the fundamental paper [BSe], Borel and Serre have shown how to enlarge X to a manifold $\bar{X}$ with "corners" on which Γ still operates properly, and for which $\bar{X}/\Gamma$ is a *compact* $K(\Gamma, 1)$-manifold with corners. The boundary is explicit enough (it has the homotopy type of a bouquet of $(l-1)$-spheres, where $l = \mathbf{Q}$-rank of G), and one derives

$$cd\Gamma = \dim X - l$$

([BSe], 11.4.3). Further, Γ is a duality group, and Poincaré if and only if $l = 0$, that is, $X(\Gamma)$ is compact.

We apply this to G = norm-1-elements of $A^\times$ and SG = elements of reduced norm 1 over the center. Then

$$\begin{aligned} \dim X(\Gamma) &= r(A), \\ \dim X(S\Gamma) &= r(SA) . \end{aligned}$$

The **Q**-rank equals $n - 1$ in both cases (see [BT], 6.21. The field- and skew field part contribute nothing to the rank.) So we see from the general theory that $X(\Gamma)$ is compact if and only if $A = D$ is a skewfield. The "if" part is our Theorem 1, and for the "only if" a direct argument is available: taking $A = M_2(\mathbf{Q})$ for simplicity it is not difficult to see that the points of X represented by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \in \mathbf{Q}^{\times}$$

cannot be uniformly bounded by right multiplication with $SL_2(\mathbf{Z})$. Summarizing, we obtain

THEOREM 4. *Let* Γ *be a unit group. Then*

$$\begin{aligned} vcd\,\Gamma &= r(A) - n + 1, \\ vcd\,S\Gamma &= r(SA) - n + 1\,. \end{aligned}$$

Γ *is a virtual duality group, and Poincaré if and only if* $n = 1$.

This is our second generalization of Dirichlet's unit theorem. Recall that the easy part of this theorem is

$$\Gamma/\text{torsion} \cong \mathbf{Z}^r,\ r \leqslant r(K)\,,$$

and the hard part is to show that $r = r(K)$. But from the Künneth formula one easily derives $cd\,\mathbf{Z}^r = r$. Another interesting consequence is

COROLLARY. Γ *contains a free subgroup of finite index if and only if* $A = M_2(\mathbf{Q})$.

Proof. In view of the theorem and formula (5) we have to show that the equation

$$r_1' \frac{(ns+2)\,(ns-1)}{2} + r_1'' \frac{(ns-2)\,(ns+1)}{2} + r_2(ns-1)\,(ns+1) = n$$

admits as only solution $n = 2$, $r_1' = s = 1$, $r_1'' = r_2 = 0$. First, we must have $r_2 = 0$ and next $r_1' r_1'' = 0$ because otherwise there would be two summands $> \frac{n}{2}$. The reader can work out that

$$r_1''(ns-2)\,(ns+1) = 2n$$

has no solution, the remaining equation only the one stated above.

In other words, $SL_2(\mathbf{Z})$ is not virtually isomorphic to any other unit group, and is virtually free. This latter property is usually proved by applying the Kurosh subgroup theorem to $PSL_2(\mathbf{Z}) = C_2 * C_3$.

The actual calculation of the integral cohomology $H^*(\Gamma)$ is hard. Being satisfied with virtual results, we can consider the case $\Gamma = SL_2(\mathbf{Z})$ as settled (if you don't, apply the Mayer-Vietoris sequence ([Se 3], 1.3) to $\Gamma = C_4 *_{C_2} C_6$.) The next case $\Gamma = SL_3(\mathbf{Z})$ requires already substantial work; the interested reader is referred to [So], and, for congruence groups, to [LS].

In view of Theorem 4, it is natural to ask for a classifying space for Γ of the "correct" dimension $vcd\,\Gamma$. Such a space has been constructed (for unit groups Γ) by Ash [Ash] as a deformation retract of the space of forms which was the object of reduction theory. Ash's construction, which is as elementary as ingenious, generalizes ideas pursued as early as 1907 by Voronoi; see [Br], ch. VIII for a discussion. Thereby the general but very involved construction of Borel-Serre can be avoided in the present case. Let us sketch the procedure in the simplest case $\Gamma = SL_n(\mathbf{Z})$: the space of forms is

$$H^+ = SO(n)\backslash SL_n(\mathbf{R}) ,$$

and what we eventually want, is a compact deformation retract of

$$H^+/SL_n(\mathbf{Z}) = SO(n)\backslash SL_n(\mathbf{R})/SL_n(\mathbf{Z}) .$$

Now

$$SL_n(\mathbf{R})/SL_n(\mathbf{Z}) =: G$$

is naturally identified with a space of lattices in $\mathbf{R}^n$; so instead of working with forms mod $SL_n(\mathbf{Z})$, we can work with lattices mod $SO(n)$. For $L \in G$ define

$$m(L) = \min\{\langle x, x\rangle \mid x \in L\backslash(0)\}$$

and

$$M(L) = \{x \in L \mid \langle x, x\rangle = m(L)\} ,$$

the set of "minimal vectors" of L. Ash calls L "well rounded" if $M(L)$ contains a basis of $\mathbf{R}^n$. It is clear that these definitions descend to $SO(n)\backslash G$. Ash's main result is that the space $W = \{$well-rounded lattices with $m(L) = 1\}$ mod $SO(n)$ is the required deformation retract.

Returning to the Dirichlet unit theorem once more, we observe that the rank $r(K)$ is detected by the cohomology in still another way. Let us work with the full unit group and write $\Gamma = C_k \times \mathbf{Z}^r$. Using the well-known cohomology of cyclic groups and the Künneth formula, one readily computes

$$H^n(\Gamma) = H^n(\Gamma, \mathbf{Z}) = \mathbf{Z}^{\binom{r}{n}} \times (C_k)^{\binom{r}{n-2} + \binom{r}{n-4} + \cdots} ,$$

so that for $n > r$

$$H^n(\Gamma) = \begin{cases} (C_k)^{\binom{r}{0} + \binom{r}{2} + \cdots + \binom{r}{r'}}, & n \text{ even} \\ (C_k)^{\binom{r}{1} + \binom{r}{3} + \cdots + \binom{r}{r''}}, & n \text{ odd}, \end{cases}$$

where $r' = r$ for r even and $r' = r - 1$ for r odd, and vice versa for r''. Thus r is recovered from $H^*(\Gamma)$ in two ways:

(1) $r = \max\{n \mid rkH^n(\Gamma) > 0\}$,

(2) $r + 1 =$ lowest dimension from which on $H^*(\Gamma)$ is periodic (of period two).

The periodicity has been generalized by Venkov [Ve] to orders in skew fields. He proves the following general theorem: Let G be a connected noncompact Lie group and $\Gamma < G$ a discrete subgroup with the properties

(i) *every finite subgroup of* Γ *has cohomological period* g;

(ii) *there is* $c \in H^g(\Gamma)$ *such that, for every finite subgroup* $H < \Gamma$, $res_H^\Gamma c$ *generates* $H^g(H)$.

Then the cup product by c *gives isomorphisms*

$$H^k(\Gamma, M) \cong H^{k+g}(\Gamma, M)$$

for all Γ*-modules* M *and* $k > \dim G - \dim C, C$ *a maximal compact subgroup.*

This too can be applied to $G =$ norm-1-group of $A^\times$, where $A = D$ has to be a skew field. G is noncompact unless D is a totally definite quaternion algebra, a case which we can happily omit from our considerations because in this case $S\Gamma$ is finite. The possible finite subgroups of $D^\times$ have been classified by Amitsur [Am]. As his results show, their Sylow groups are cyclic or generalized quaternion groups; hence they have periodic cohomology ([Br], th. VI 9.5). Since Γ contains — up to isomorphism — only finitely many finite subgroups, these have a common period. These arguments are not even necessary because Venkov shows ([Ve] Prop. 5) that, for $g = \dim_Q D$,

$$H^g(H) = \mathbf{Z}/|H|\mathbf{Z}$$

for any finite subgroup H of Γ; this implies g-periodicity by ([Br], Th. VI 9.1). The harder part is condition (ii) which requires a spectral sequence argument. One obtains

THEOREM 5. *Suppose that* $A = D$ *is not a totally definite quaternion algebra. Then there are isomorphisms*

$$H^k(\Gamma, M) \cong H^{k+g}(\Gamma, M)$$

for all Γ*-modules and* M *and* $k > r(D)$. *Moreover,* $r(D) + 1$ *is the smallest dimension from which on* $H^*(\Gamma, -)$ *is n-periodic.*

The last statement follows from the facts that (1) Γ is a virtual duality group and hence $H^{r(D)}(\Gamma, \mathbf{Z}\Gamma)$ is torsion free ([Br], VIII 11.2) whereas (2) for any group Γ with $vcd\,\Gamma = k$, $H^m(\Gamma)$ is torsion for $m > k$ ([Se 3], p. 101).

It should be possible to refine the period g in the theorem by one more directly derived from the periods of the finite subgroups of Γ as in the number field case where the period equals 2 if there are nontrivial torsion units of norm 1 (in which case n must be even!).

REMARK. We have seen (in the general case, Γ torsionfree) that

$$H^*(\Gamma, -) = H^*(X(\Gamma), -)\ .$$

Taking real coefficients (with trivial Γ-action) the latter groups are, by de Rham's theorem, given by differential forms on $X(\Gamma)$; these in turn correspond to Γ-automorphic forms on X. In this way, the real cohomology of Γ becomes part of the theory of automorphic forms.

8. CONGRUENCE SUBGROUPS AND NORMAL SUBGROUPS

Recall that we have defined

$$\Gamma(m) = \text{kernel of } (\Gamma \to (\Lambda/m\Lambda)^\times)\ ,$$

the congruence subgroup of level m of Γ. Obviously $\Gamma(m)$ has finite index in Γ. The following question is classical: does every subgroup of finite index of Γ contain a congruence group?

Let us say that Γ satisfies (CP) if this is so. Let $\Lambda \subset \Lambda'$. If Γ' satisfies (CP), so does Γ. To prove the converse, it suffices to show that every $\Gamma(n)$ contains a $\Gamma'(m)$. This will be so if Γ contains a $\Gamma'(m)$. But there is $m \in \mathbf{N}$ with $m\Lambda' \subset \Lambda$, and this implies $\Gamma'(m) \subset \Lambda \cap \Gamma' = \Gamma$. Thus, property (CP) depends only on A.

For $A = K$ a number field, (CP) has essentially been proved by Chevalley [Ch]. Let $H < R^\times$ be of finite index, and $H_0 < R^\times$ any congruence subgroup. Then $H_0^k \subset H$ for some $k \in \mathbf{N}$; so it suffices to show that any power of a congruence group contains a congruence group. This follows at once from ([Ch], Th. 1). By an argument presented earlier, this allows us to reduce the problem to $S\Gamma$. If $D = K$, the answer is given by results of Bass-Milnor-Serre [BMS] and Serre [Se 5]. (Of course, one defines congruence groups with respect to ideals of R. For (CP) to hold, this makes no difference since every ideal of R contains an nR, $n \in \mathbf{Z}$.)

THEOREM 6. *Assume* $D = K$.

(a) [BMS] *If* $n \geqslant 3$, $SL_n(R)$ *has property (CP) if and only if* K *has a real embedding.*

(b) [Se 5] $SL_2(R)$ *has property (CP) if and only if* $r(K) > 0$.

The only K with $r(K) = 0$ are $K = \mathbf{Q}$ and $K =$ imaginary quadratic. The failure of (CP) for $SL_2(\mathbf{Z})$ is classic (see [Ne], p. 149). A discussion of the Bianchi case is given in [F, p. 200 ff.].

The case where $K \neq D$ and $n > 1$ was treated by Bak and Rehmann [BR]. They give an S-arithmetical result, from which we extract:

THEOREM 7. *Suppose* $K \neq D$ *and* $n \geqslant 2$; *if* $n = 2$ *suppose* $K \neq \mathbf{Q}$ *and that* D *is not a definite quaternion algebra. Then (CP) holds.*

The remaining cases seem still to be unsettled. Note that in the definite quaternion case the problem becomes trivial since $S\Gamma$ is finite.

The congruence groups are special since normal subgroups having finite index. It also makes good sense to ask whether every noncentral normal subgroup of $S\Gamma$ has finite index. This too may be asked more generally for discrete subgroups of Lie groups, and definite results have been obtained by Margulis. We specialize these to the case in question. As above, we can give no details of the proofs, which are extremely involved, and refer the reader to Margulis' monumental volume [M]. We make the following assumptions, excluding $A = K$:

$$(H) \quad \begin{cases} \text{(i) if } n = 1, \text{ then } D \text{ splits at all infinite primes of } K; \\ \text{(ii) if } K = \mathbf{Q}, \text{ then } k \geqslant 3, \text{ and } A \neq M_2(D),\ D \text{ definite quaternion.} \end{cases}$$

This means that the norm-1-group SG has no anisotropic factor and has rank $\geqslant 2$. It is clear that — in the terminology of [M] — $S\Gamma$ is an irreducible lattice of SG. Applying Theorems (2) and (4) from the introduction of [M], we obtain

THEOREM 8. *Assume* (H). *Then every noncentral normal subgroup of* $S\Gamma$ *has finite index.*

COROLLARY. $S\Gamma/[S\Gamma, S\Gamma]$ *is finite.*

Again, it is classic that the theorem fails for $SL_2(\mathbf{Z})$; for instance, if $n \gg 0$, the verbal subgroup generated by all $6n$-th powers has infinite index ([Ne], p. 143). The corollary holds for $SL_2(\mathbf{Z})$ but fails for torsion free subgroups (which are free). Theorem 7 however carries over to subgroups of finite index.

Two more topics from the general theory of arithmetic groups, which could be specialized to unit groups, are subgroup rigidity and strong approximation. But having promised to keep as near as possible to the unit groups "themselves", we omit this.

9. THE BASS UNIT THEOREM

In his paper [Ba 2] Bass has proved (among other things) a far reaching generalization of Dirichlet's unit theorem which — together with the results of sections 3 and 7 — is certainly one of the strongest general results we have about Γ. The core of the proof is a deep stability theorem from K-theory; we will indicate how it implies the theorem but will say little about its proof. We begin with the relevant definitions. For any ring A, define

$$K_1(A) = \varinjlim GL_n(A)/[GL_n(A), GL_n(A)] \ ,$$

where the direct limit is taken with respect to the embeddings

$$GL_n(A) \to GL_{n+1}(A), \quad x \to \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} .$$

One may also write

$$K_1(A) = \varinjlim \quad GL_n(A)/\tilde{E}_n(A) \ ,$$

where $\tilde{E}_n(A)$ is the normal subgroup generated by the elementary matrices; this is Whitehead's lemma. Further, with $K_0(A)$ denoting the Grothendieck group of finitely generated projective A-modules, we put

$$R_0(A) = \mathbf{R} \otimes K_0(A), \ R_1(A) = \mathbf{R} \otimes K_1(A) \ .$$

Now we turn to algebras and allow A to be semisimple. Let $\Lambda \subset A$ be an order. Any $A_{\mathbf{R}}$-module V (of finite dimension) gives rise to a homomorphism

$$\Lambda \to \mathrm{End}_{\mathbf{R}} V, \text{ hence by functoriality } K_1(\Lambda) \to K_1(\mathrm{End}_{\mathbf{R}} V) \ .$$

Combining this with

$$\log|\det| : K_1(\mathrm{End}_{\mathbf{R}} V) \to \mathbf{R} ,$$

we obtain a homomorphism

$$f_V : R_1(\Lambda) \to \mathbf{R} ,$$

and it is easy to see that $V \to f_V$ gives a homomorphism

$$f : K_0(A_{\mathbf{R}}) \to R_1(\Lambda)^* = \mathrm{Hom}(R_1(\Lambda), \mathbf{R}) .$$

THEOREM 9. *The sequence*

$$0 \to R_0(A) \to R_0(A_{\mathbf{R}}) \to R_1(\Lambda)^* \to 0 \tag{8}$$

is exact.

The following corollary (which can easily be derived from the sequence) is shown in the course of the proof:

COROLLARY. *Let R be the maximal order of the center K of A. Then*

$$rk K_1(\Lambda) = rk R^\times .$$

Dirichlet's theorem arises in the special case $\Lambda = R$, $A = K$. Clearly $rk K_0(K) = 1$, and writing

$$\mathbf{R} \otimes_{\mathbf{Q}} K = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$$

as previously, we obtain

$$rk K_1(R) = r_1 + r_2 - 1 .$$

But for Dedekind domains R one knows that

$$rk R^\times = rk K_1(R)$$

(see [CR 2], §45 A). However, Dirichlet's theorem is used in the proof of Theorem 7.

A case of interest is

$$A = \mathbf{Q}G, \Lambda = \mathbf{Z}G \quad (G \text{ is a finite group}) .$$

Here,

$$\begin{aligned} rk K_0(\mathbf{Q}G) &= \text{number of conjugacy classes of cyclic subgroups of } G \\ &= : q(G), \\ rk K_0(\mathbf{R}G) &= \text{number of real conjugacy classes of } G \\ &= : r(g) \end{aligned}$$

(see e.g. [Se 6], 12.4). Thus the theorem gives

$$rk K_1(\mathbf{R}G) = r(G) - q(G) .$$

By Theorem 5 of [Ba 2], $(\mathbf{Z}G)^\times$ is mapped onto a subgroup of finite index in $K_1(\mathbf{Z}G)$. Hence

$$rk\left(\frac{\mathbf{Z}G^\times}{[\mathbf{Z}G^\times, \mathbf{Z}G^\times]}\right) \geqslant r(G) - q(G) .$$

The reader will find it an amusing exercise to work out that for G cyclic this is an equality. However, if $\mathbf{Q}$ is a splitting field for G (as e.g. for G = symmetric group) the inequality tells us nothing new.

Proof of the theorem (sketch). The injectivity of $K_0(A) \to K_0(A_\mathbf{R})$ follows from a wellknown theorem of representation theory (see [CR 1], § 29). Next, it is not difficult to show that $R_0(A) \to R_0(A_\mathbf{R}) \to R_1(\Lambda)^*$ is the zero map: if V is already an A-module, then

$$K_1(\Lambda) \to K_1(\mathrm{End}_\mathbf{R} V) \to \mathbf{R}$$

factors through

$$K_1(\Lambda) \to K_1(\mathrm{End}_\mathbf{Q} V) \to \mathbf{R} ;$$

if $x \in K_1(\Lambda)$ is represented by an element of $Gl_n(\Lambda)$, the image of this element in $K_1(\mathrm{End}_\mathbf{Q} V)$ is an integral unit in a matrix ring over $\mathbf{Q}$, hence has determinant $\pm$ 1. In order to state the crucial lemma, we recall the notion of reduced norm. Assume for the moment that A is simple with center K. Then there exists a splitting field $L \mid K$, for example, a maximal commutative subfield of A, such that

$$L \otimes_K A \cong M_n(L)$$

is isomorphic to a full matrix algebra. Let φ be an isomorphism and define, for $a \in A$,

$$Nr(a) = \det \varphi(1 \otimes a) .$$

One shows that $Nr(a) \in K$ and is independent of the choice of L and φ. The effect of using Nr instead of the usual norm $N_{A \mid K}$ taken with respect to the regular representation of A over K is the elimination of superflous powers; namely, one has

$$N_{A \mid K}(a) = (Nr\,a)^s, \quad \text{where } \dim_K A = s^2 .$$

For semisimple A, Nr is defined componentwise and induces homomorphisms

$$Nr_n : GL_n(\Lambda)/\tilde{E}_n(\Lambda) \to R^\times$$

and in the limit

$$N : K_1(\Lambda) \to R^\times .$$

It is easily seen that Nr_n and N have finite cokernel. Let q be the number of simple factors of A.

LEMMA. *For n sufficiently large, Nr_n and therefore N have finite kernel. Consequently,*

$$rkK_1(\Lambda) = rkR^\times = r_1 + r_2 - q .$$

Taking this for granted, we combine N with

$$\log | \, | : R^\times \to \mathbf{R}^{r_1 + r_2}, \ c \to (\log |c|_1 \ldots, \log |c|_{r_1 + r_2}) .$$

From the lemma, one first derives that every component of

$$\log |N| : K_1(\Lambda) \to \mathbf{R}^{r_1 + r_2}$$

has the form f_V for suitable V. Then it follows that there are "enough" linear functionals of this type, that is,

$$f : R_0(A_{\mathbf{R}}) \to R_1(\Lambda)^*$$

is surjective. The exactness of (8) now follows by dimension count.

We cannot say much about the proof of the lemma and refer the reader to [Ba 1]. The main point is that 2 defines a "stable range" for the $\mathbf{Z}$-algebra Λ ([Ba 1], Th. 11.1) which implies, among other things, that for $r > 2$

$$GL_r(\Lambda) = GL_2(\Lambda)E_r(\Lambda)$$

(hence $GL_2(\Lambda) \to K_1(\Lambda)$ surjective) and for $r \geqslant 4$

$$E_r(\Lambda) = [GL_r(\Lambda), GL_r(\Lambda)] .$$

([Ba 1], th. 19.5). Put $Sl_n(\Lambda)$ = Kernel of reduced norm. The above implies that for $n \geqslant 4$ all maps

$$SL_n(\Lambda)/E_n(\Lambda) \to SL_{n+1}(\Lambda)/E_{n+1}(\Lambda) \to SL(\Lambda)/E(\Lambda)$$

are surjective and all these groups are abelian. Since everything is finitely generated, this sequence becomes stationary, i.e.

$$SL_n(\Lambda)/E_n(\Lambda) \cong SL(\Lambda)/E(\Lambda)$$

for all $n \gg 0$. One then has an exact sequence

$$0 \to SL(\Lambda)/E(\Lambda) \to K_1(\Lambda) \to K_1(A) \ ,$$

and it remains to show that $K_1(\Lambda) \to K_1(A)$ has finite kernel ([Ba 1], 19.12). This implies the lemma.

We have presented Bass' theorem here because it can be viewed as an extension of Dirichlet's unit theorem. For more results on K_1 of orders, we refer the reader to [CR 2, Ch. 5]. This chapter also contains a simplified proof of Bass' theorem.

10. What is a unit theorem?

In the search for the — still missing — "basic structure theorem for units of orders" it seems natural to keep Dirichlet's theorem as our landmark; it gives in fact a presentation for all commutative unit groups. However, if we muster the small list of other cases in which explicit presentations have been obtained so far, and if we realize the comparatively elementary character of these examples, we have to admit that going for presentations is somehow utopian. Worse still, it might even be inadequate; as the general insolvability of Dehn's problems shows, we can never be sure that a presentation, obtained somehow, gives us the "right" information. For example, how could the congruence property be checked from a presentation? What then, it will now be objected, is the aim of our research? This is certainly not the place to dwell in considerations in the manner of ordinary language philosophy, but the reader may find it fruitful to ask himself what he means by saying "I know a certain group" or "I know the structure of that group". Surely we know $SL_2(\mathbf{Z})$ better than any other noncommutative unit group, but we will never know everything about it (and hence about groups containing it) because this would include knowledge of all finitely generated groups.

Leaving aside philosophy, let us try to specify what should be expected from a "general unit theorem". Unable, of course, to presume its content, we may be allowed to sketch a list of desiderata.

Let A be simple. The unit theorem should deal with torsion free subgroups of finite index of $S\Gamma$ for arbitrary Λ; such groups may be called "*generic unit groups of* A". The set of generic unit groups is closed under intersections since any two are commensurable. Naively, a unit theorem for A consists in *the definition, in purely group theoretical terms, of a class of groups* $\mathscr{C}(A)$ *such that almost all generic unit groups of* A *are members of* $\mathscr{C}(A)$.

Of course the elements of $\mathscr{C}(A)$ must have all the properties we have established for the $S\Gamma$; in particular they must be finitely presented and of cohomological dimension $r(SA) - n + 1$. They should be parametrized by the numerical invariants of A plus a parameter accounting for the index. By numerical invariants I mean the various degrees involved as well as $r(SA)$, discr Λ for Λ maximal, perhaps even class numbers and Hasse invariants. The smaller $\mathscr{C}(A)$ the better the unit theorem; optimally, $\mathscr{C}(A)$ consists — perhaps up to finitely many exceptions — of the generic unit groups of A. Our two examples are $A = M_2(\mathbf{Q})$, in which $\mathscr{C}(A)$ consists of the finitely generated free groups, and A = indefinite quaternion skewfield over $\mathbf{Q}$, in which $\mathscr{C}(A)$ consists of the fundamental groups of closed oriented surfaces.

One should realize that the existence of a definition of $\mathscr{C}(A)$ independent of A is in no way guaranteed, in other words, that there may be no pre-existing group theoretical terms by which the generic unit groups of A can be characterized. This would mean that there are algebras (presumably skewfields) which produce group — theoretical features not available from anywhere else, at least not with lesser complexity. The simplicity of the examples is surely misleading. But this may be a question for logicians and complexity theorists rather than for an "ordinary" mathematician.

Given A, we would like to distinguish in $\mathscr{C}(A)$ the maximal generic unit groups. For $A = M_2(\mathbf{Q})$, one is a free group of rank 2, occurring as the commutator group of $SL_2(\mathbf{Z})$. (I don't know whether or not all maximal torsion free subgroups of $SL_2(\mathbf{Z})$ are free of rank 2).

Given A_1 and A_2, we would like to decide whether or not they share a generic unit group (and hence infinitely many). In the number field case the unit rank is a rather weak invariant. In contrast to this, $SL_2(\mathbf{Z})$ is unique, as we have seen in section 7. In the quaternion case, there are coincidences (see the end of [E 1]).

Traditionally the geometry connected with the unit groups was considered more important than the groups themselves. Paying tribute to this view we could formulate geometric analoques to the above questions. Let SG be the elements of $A_{\mathbf{R}}^{\times}$ of reduced norm one, $C \subset SG$ a maximal compact subgroup. For generic $\Delta \subset SG$ put

$$X(\Delta) = C \backslash SG / \Delta \, .$$

Then the overall program would be *to study the manifolds* $X(\Delta)$. This is surely the most ambitious part, and pointing to the vast amount of work which has been and is currently devoted to the simplest non-settled $S\Gamma$, the Hilbert modular groups, one might criticize this laconic formulation as all too naive. (The reader who wants to get an impression of the world of mathematics meeting here should have a glance to the volume [Ge]). On the other hand, being content with subgroups of finite index, we avoid the complications arising from the torsion in $S\Gamma$. It is also conceivable that the projective system of all $X(\Delta)$ and its limit is the appropriate subject of our hypothetical "basic unit theorem". Again it can be asked what is meant by "knowing a space". A "space presentation", as analogue to a group presentation, could be an explicit cell structure; this has been obtained in a few cases. But here as elsewhere in mathematics one cannot hope to get "everything explicit"; the real problem is to define the significant invariants and to understand their mutual relations. If there is a single theorem deserving the name "General Unit Theorem" it will probably relate arithmetical and geometrical invariants.

Of particular importance will be those of cohomological origin. Note that in our two examples the decisive invariant (rank and genus, respectively) is nothing more than a first Betti number. It is clear that things wont't be so easy generally. But at least the following result deserves to be mentioned here: for generic unit groups $\Delta_1 \subset \Delta_2$ one has

$$|\Delta_2 : \Delta_1| \chi(\Delta_2) = \chi(\Delta_1) ,$$

χ denoting Euler characteristics. (See [Se3], p. 86). If these don't vanish, this is an index formula, generalizing the Nielsen-Schreier formula

$$|\Delta_2 : \Delta_1| (rk\Delta_2 - 1) = (rk\Delta_1 - 1)$$

for $A = M_2(\mathbf{Q})$ and the Riemann-Hurwitz formula

$$|\Delta_2 : \Delta_1| (g(\Delta_2) - 1) = (g(\Delta_1) - 1)$$

in the quaternion case (g denoting genus).

Finally, let us muster the algebras with small $r(SA)$ and see what could be done next. We exclude $A = K$; that is $k = ns > 1$ formula (5). Note that $r_1'' > 0$ implies s even, in particular > 0. $r(SA) = 0$ occurs only for $r_1' = r_2 = 0$, $n = 1$, $s = 2$, r_1'' arbitrary. This means that A is a totally definite quaternion skew field, and we have noted already that $S\Gamma$ is finite in such cases which we therefore consider as settled. (It is interesting to note that these algebras are exceptional in other respects, too — to "compensate" for the easy unit theory, their module theory is more difficult.) $r(SA) = 1$ is not possible (as the reader should check from (5). (Conceptual explanation: if $r(SA) = 1$, then a generic unit group would be the fundamental group of a one dimensional manifold, hence abelian. On the other hand, if it is infinite, it is Zariski dense in SG, by a theorem of Borel ([P]), Th. 1.5). Thus, A would be commutative). If $r(SA) = 2$, by necessity $r_2 = 0$, $ns = 2$, $r_1' > 0$. We may have $n = 2$, $s = 1$ and consequently $r_1'' = 0$; this gives $A = M_2(\mathbf{Q})$; or $n = 1 = r_1'$, $s = 2$ and r_1'' arbitrary. Then A is a quaternion skew field over a totally real K ramified at all but one of the infinite primes of K. (Eichler's case is $r_1'' = 0$.) The image of the $S\Gamma$ in $PSL_2(\mathbf{R})$ are special Fuchsian groups characterized among all Fuchsian groups by the behavior of their traces [Ta]. Now finitely generated Fuchsian groups have a standard presentation given by their "signature" (see [F], p. 37). It should be possible to calculate the signatures in terms of the arithmetic invariants of A, generalizing Eichler's result. $r(SA) = 3$ requires $r_1' = 0$, $r_2 = 1$, $ns = 2$, r_1'' arbitrary. $s = 1$, $r_1'' = 0$ is the case of the Bianchi groups. For $n = 1$, $s = 2$, r_1'' arbitrary A is a quaternion skewfield over a field K with one complex embedding, ramified at all real infinite primes of K. The images of $S\Gamma$ in $PSL_2[\mathbf{C}]$ are special Kleinian groups, acting discontinuously on hyperbolic 3-space. It should be possible to treat them as the Bianchi groups. Similarly with $r(SA) = 4$ we encounter the Hilbert modular groups, but also quaternion skewfields over totally real fields ramified at all but two of the infinite primes ($r_2 = 0$, $r_1' = 1$, $s = 2$, $n = 1$, r_1'' arbitrary). At least if $r_1'' = 0$ (so A is ramified only at finite primes) the skewfield case can hardly be of more complicated structure than the matrix case; it should be even easier in view of the fact that bounded fundamental domains exist. That they have been studied much less must probably be ascribed to the circumstance that it is not so easy to write down units in skewfields. This brings us to our last point namely the

PROBLEM. *Give an algorithm which constructs generators of a subgroup of finite index of* $S\Gamma$.

This problem has in principle been solved by Grunewald and Segal ([GS], Algorithm B). As so many other results of this survey, their algorithm applies to arithmetic groups and is, as the authors point out, even in this generality not best possible. Bringing in, in the case of units of orders, the underlying ring structure, one should be able to give manageable procedures. The main interest lies in the case $A = D$ which seems to be untouched (in this respect). Since every $x \in \Gamma^\times \setminus R^\times$ generates an extension of number fields $K(x) \mid K$, the methods of computational number theory will enter the game. In view of this, it will be of advantage that we may choose Λ to be a cyclic crossed product order.

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The following book came too late for this essay but should be mentioned here because it surely will become a standard text on arithmetic groups:

V. PLATONOV, A. RAPINCHUK. *Algebraic Groups and Number Theory.* Academic Press, 1994.

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