2. Background and Definitions

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2. Background and Definitions

Let X be a topological space and f a homeomorphism of X. We say that f is periodic if there is an integer $n > 0$ such that $f^n = Id$. The period of f is the smallest positive integer n with this property.

As we will use them without further justifications, let us first recall some basic properties of one-dimensional maps.

Let $f: I \to I$ be a periodic homeomorphism of the unit interval. If f preserves the endpoints then f is the identity map. If f exchanges the endpoints then $f^2 = Id$ and f is conjugate to the reflection map $x \mapsto 1 - x$. Similarly, a periodic homeomorphism of the real line $\mathbf{R}$ is the identity map or is a conjugate of the involution $x \mapsto -x$ according to whether it is an increasing or a decreasing function.

Let $f: S^1 \to S^1$ be a periodic homeomorphism of period n of the unit circle. If f is order-preserving then the rotation number of f, $\rho(f) = k/n$, where k and n are coprime (see [5] for an excellent exposition on rotation numbers) and f is conjugate to a rotation of angle $2k\pi/n$. If f is order-reversing then f has exactly two fixed points, f^2 is the identity map and the two arcs delimited on S^1 by the fixed points of f are permuted by f.

A metric space X is path connected if there exists a continuous map from the unit interval [0, 1] into X which joins any two given points. It is arcwise connected if there is a topological embedding of [0, 1] into X which joins any two given distinct points. In fact, it can be shown that the two notions are equivalent (see [14, Theorem 4.1] or [11, Lemma 16.3]).

Lemma 2.1. *A metric space X is path connected if and only if it is arcwise connected.*

A useful characterisation of path connected spaces is given in term of local connectivity. A metric space X is locally connected if each point of X possesses arbitrary small connected neighbourhoods. The following can be shown (see [8, Theorem 3.15] or [11, Lemma 16.4]):

Lemma 2.2. *A compact, connected and locally connected metric space is pathwise connected.*

Another important ingredient used in this article, and in fact the ultimate result we will need, is the famous Jordan-Schoenflies theorem on simple closed curves in the plane (see [2, 9] or [12, Theorem 17.1]).

Theorem 2.3 (Jordan-Schoenflies). *Every simple closed curve J divides the plane into exactly two components of each of which it is the* *complete boundary and the closure of the bounded component can be mapped topologically onto the closed unit disc.*

In what follows, a *closed* topological disc (or just a topological disc) D is the image under a topological embedding of the *closed* unit disc and we write D^o for its interior and ∂D for its boundary. However, the closure of a bounded open set which is homeomorphic to the open unit disc is not necessarily a closed topological disc [11, Chapter 15].

PROPOSITION 2.4. *Let* $D_1, D_2, \ldots, D_n$ *be a finite number of closed topological discs in the plane and* J^o *be any connected component of* $\bigcap_{i=1}^{n} D_i^o$. *Then* ∂J *is a simple closed curve and* J *the closure of* J^o *is a topological disc.*

Proof of 2.4. We will use induction on n, the number of discs. If $n = 1$ this is just the Jordan-Schoenflies theorem, so let us suppose that the result holds for some $n (n \geqslant 1)$ and let J^o be any component of the complement of $n + 1$ topological discs $D_1, D_2, \ldots, D_{n+1}$ in the plane. Let K^o be the component of $\bigcap_{i=1}^{n} D_i^o$ that contains J^o. By induction, its closure K is a topological disc. Since J^o is a component of $K^o \cap D_{n+1}^o$, it suffices to show that the result holds for two discs D_1 and D_2 (see Figure 1). Set $C_i = \partial D_i$ for $i = 1, 2$ and let J be the closure of a component of $D_1^o \cap D_2^o$. We have that $\partial J \neq \emptyset$ and $\partial J \subset C_1 \cup C_2$. If ∂J is entirely contained in one of the two curves, say C_1, then $J = D_1$ and the lemma is proved. We can thus suppose that $\partial J \not\subset C_1$ and $\partial J \not\subset C_2$.

Let $x \in \partial J$, $x \notin C_2$. Then $x \in C_1 \cap D_2^o$, and we can find an arc γ in C_1 such that:

$$x \in \gamma\,, \quad \gamma \subset \partial J\,, \quad \gamma \backslash \partial\gamma \subset D_2^o, \quad \partial\gamma \subset C_2\,.$$

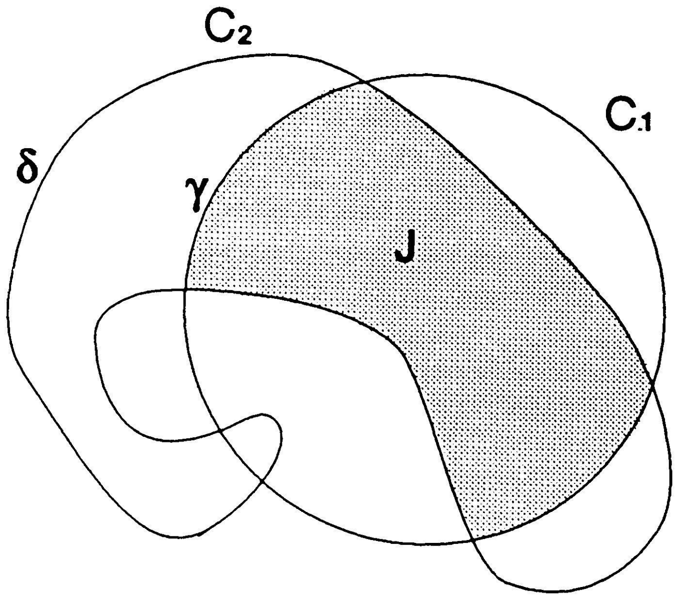

FIGURE 1

The endpoints of γ determine on C_2 an arc δ disjoint from J^o and such that $\delta \cap J = \partial\delta$. We note that there is an at most countable family of such arcs γ, noted $(\gamma_i)_{i \in N}$ and that $diam(\gamma_i) \to 0$ as $i \to \infty$. The boundary of J is the simple closed curve obtained from C_2 when substituting the arcs γ_i for the arcs δ_i and J is a topological disc by the Jordan-Schoenflies theorem. □

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane $\mathbf{R}^2$, namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let $f : S \to S$ be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold S and let $x \in Fix(f)$, a fixed point of f. Then for any neighbourhood N of x, there exists a topological disc Δ_x such that:*

1. $\Delta_x \subset N$,

2. Δ_x *is a neighbourhood of* x,

3. $f(\Delta_x) = \Delta_x$.

Proof of 2.5. We can first assume that N and its image under f, $f(N)$, are contained in some local chart U homeomorphic with $\mathbf{R}^2$ and will continue to call x and N the corresponding point and set in $\mathbf{R}^2$. Let D_x be an euclidean disc of centre x and radius η where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, .., n-1$ and let C_x be its boundary. Let Δ_x be the closure of the component of the invariant set $\cap_{k=0}^{n-1} f^k(D_x^o)$ which contains x. By 2.4, Δ_x is a topological disc which is invariant under f (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. □

Remark. The boundary γ_x of Δ_x, which is an invariant simple closed curve, is contained in $\cup_{k=0}^{n-1} f^k(C_x)$.

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THEOREM 3.1. *Let $f : D^2 \to D^2$ be a periodic homeomorphism. Then there exists $r \in O(2)$ and a homeomorphism $h : D^2 \to D^2$ such that $f = hrh^{-1}$.*