6. The number of proper 4-colorings of a graph

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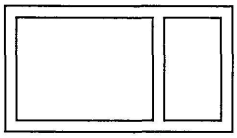
For example, all elements of weight 3 in $K_{m,n}$ can be represented (up to proper size and location) by the following picture:



or its vertical analogue. This picture represents a codeword of the form

$$E(k, l; r_1, r_2) + E(k, l; r_1, r_3) + E(k, l; r_2, r_3) .$$

Thus, the number of codewords of weight 3 in $K_{m,n}$ is equal to

$$w_3(K_{m,n}) = \binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2} .$$

Similarly, one can show that

$$w_4(K_{m,n}) = 3\binom{m}{2}\binom{n}{4} + 9\binom{m}{3}\binom{n}{3} + 3\binom{m}{4}\binom{n}{2} ;$$

$$w_5(K_{m,n}) = 12\binom{m}{2}\binom{n}{5} + 72\binom{m}{3}\binom{n}{4} + 72\binom{m}{4}\binom{n}{3} + 12\binom{m}{2}\binom{n}{5} + 9\binom{m}{3}\binom{n}{3} .$$

As a last remark, note that an upper bound for the weights in the associated code L_f is given by $\frac{1}{8}n^3(n-1)$, and that this bound is actually attained for some n if and only if there exists a Hadamard matrix of order n. This follows from, say, Corollary 3.

6. The number of proper 4-colorings of a graph

Let $G = (V, E)$ be a simple graph (no loops, no multiple edges) with vertex set V and edge set E. We will identify V with $\{1, \ldots, n\}$, and denote the cardinality of E by e.

A 4-*coloring* of G is the assignment to every vertex of one among four fixed colors; such a coloring is *proper* if the colors assigned to the end vertices of any edge are distinct. For a survey on the 4-colorings of planar graphs, see [SK].

We will count the number of proper 4-colorings of G, in terms of the weight enumerator of a certain code of length $3e$.

Step 1. *The defining equations for proper 4-colorings.*

As our palette of colors, we will choose the 4-set $\{1, -1\}^2$. The space of all 4-colorings of G can thus be identified with $\{1, -1\}^{2n}$, for example as follows:

Convention. If $p = (p_1, \ldots, p_{2n}) \in \{1, -1\}^{2n}$, then the pair (p_i, p_{i+n}) represents the color assigned to vertex i, for $i = 1, \ldots, n$.

Consider now $2n$ variables $x_1, \ldots, x_{2n}$, and define

$$g_{i,j} := (1 + x_i x_j)(1 + x_{i+n} x_{j+n})$$

for $1 \leqslant i, j \leqslant n$.

If p is a 4-coloring of G, then $g_{i,j}(p) = 0$ if and only if the colors assigned to vertices i and j are distinct.

Thus, a 4-coloring p of G is proper if and only if

$$g_{i,j}(p) = 0 \quad \text{for all } (i,j) \in E .$$

STEP 2. *Reduction to a single equation.*

Let

$$g := \sum_{(i,j) \in E} g_{i,j}^2 .$$

By construction, g satisfies the following properties:

(1) $g(p) \geqslant 0$ for all 4-colorings p;

(2) $g(p) = 0$ if and only if p is proper.

Developing the expression for g, we obtain:

$$\begin{aligned} g &= \textstyle\sum_E g_{i,j}^2 \\ &= \textstyle\sum_E (1 + x_i x_j + x_{i+n} x_{j+n} + x_i x_j x_{i+n} x_{j+n})^2 \\ &= 4 \textstyle\sum_E (1 + x_i x_j + x_{i+n} x_{j+n} + x_i x_j x_{i+n} x_{j+n}) \\ &= 4e + 4f, \end{aligned}$$

where

$$f := \sum_E (x_i x_j + x_{i+n} x_{j+n} + x_i x_j x_{i+n} x_{j+n}) .$$

(Here again, the computation was performed modulo $x_i^2 = 1$ for all i.)

Obviously, f satisfies the following properties:

(1) $f(p) \geqslant -e$ for all 4-colorings p;

(2) $f(p) = -e$ if and only if p is proper.

STEP 3. *The code associated with f.*

Let $K_G := L_f^{\perp}$ be the dual of the code L_f associated with f. To describe it, we consider the map

$$\begin{array}{rcl} \phi_G: \mathbf{F}_2^{3e} & \to & \mathbf{F}_2^{2n} \\ E_{i,j} & \mapsto & e_i + e_j \\ E_{i+n,j+n} & \mapsto & e_{i+n} + e_{j+n} \\ E_{i+2n,j+2n} & \mapsto & e_i + e_j + e_{i+n} + e_{j+n} \quad ((i,j) \in E) \end{array}$$

where $\{E_{i,j}, E_{i+n,j+n}, E_{i+2n,j+2n}\}_{(i,j)\in E}$ and $\{e_i\}_{i\in V}$ denote the standard bases of the left and right spaces, respectively. By construction, $K_G = \operatorname{Ker}(\phi_G)$.

Here again, as a direct consequence of Theorem 4 and of the stated properties of f, we have:

THEOREM 7. *Let K_G be the code of length $3e$ defined above, and let $P_G(T)$ denote its weight enumerator. Then, the number $\chi_G(4)$ of proper 4-colorings of G, is given by*

$$\chi_G(4) = \frac{1}{4^{e-n}e!} P_G^{(e)}(-1),$$

where $P_G^{(e)}(-1)$ denotes the value at -1 of the e-th derivative of $P_G(T)$.

Proof. Apply the formula of Theorem 4 by replacing

- N, the length of the code, by $3e$;
- v, a lower bound for the range of f, by $-e$;
- n, the number of variables in f, by $2n$. □

Note that there are some obvious elements of weight 3 in K_G, namely

$$E_{i,j} + E_{i+n,j+n} + E_{i+2n,j+2n} \quad ((i,j) \in E).$$

More interestingly, every cycle of length r in G gives rise to at least 3 elements of weight r in K_G. Indeed, if $(i_1, \ldots, i_r)$ is a cycle, then

$$E_{i_1+kn,\,i_2+kn} + E_{i_2+kn,\,i_3+kn} + \cdots + E_{i_r+kn,\,i_1+kn} \in K_G,$$

for $k = 0, 1, 2$.

One can go a little bit further. A somewhat technical computation shows that $P_G(T)$ can be decomposed as follows:

$$P_G(T) = (1+T^3)^e P_{\tilde{K}}\left(\frac{T}{1-T+T^2}\right),$$

where $\tilde{K}$ is the $\mathbf{F}_4$-code defined by the exact sequence

$$0 \to \tilde{K} \to \mathbf{F}_4^e \to \mathbf{F}_4^n ,$$

with weight enumerator $P_{\tilde{K}}(T)$. (The map on the right sends the basis vector corresponding to an edge, to the formal sum of its two endvertices.)

With the above formula, we find that $P_G^{(e)}(-1) = 3^e e!\ P_{\tilde{K}}\left(-\frac{1}{3}\right)$. Plugging this into Theorem 7, we obtain

$$\chi_G(4) = \frac{3^e}{4^{e-n}} P_{\tilde{K}}\left(-\frac{1}{3}\right) .$$

In particular, G is 4-colorable if and only if $P_{\tilde{K}}\left(-\frac{1}{3}\right) \neq 0$. See also [E, Theorem 5.7], for a different formulation and proof of this formula.

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