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ENUMERATIVE COMBINATORICS AND CODING THEORY

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ABSTRACT. Let f be a polynomial in n variables with non-negative integral coefficients. The enumeration of the values assumed by f on $\{1, -1\}^n$ is shown to be equivalent to the enumeration of the weights in some associated binary linear code L_f. We use this correspondence, together with the MacWilliams identity, to enumerate (1) Hadamard matrices of some fixed order, and (2) the proper 4-colorings of a graph, in terms of the weight distribution of suitable binary codes. Similar formulas could be obtained for other combinatorial objects. [2])

Let f be a polynomial in n variables with non-negative integral coefficients. We will address here the following

PROBLEM. Is there a point $p \in \{1, -1\}^n$ such that $f(p) = 0$? How many such binary zeros does f admit? More generally, what can be said about the *value enumerator* of f, which we define as

$$V_f(T) = \sum_{p \in \{\pm 1\}^n} T^{f(p)} \in \mathbf{N}[T, T^{-1}] \ ?$$

(Note that the coefficient of T^v in $V_f(T)$ is the number of binary points p such that $f(p) = v$, for $v \in \mathbf{Z}$.) Many classical combinatorial problems can be expressed in the above terms, for a suitable polynomial f.

In order to discuss these problems, we associate with f a binary linear code L_f, in such a way that the weight enumerator of L_f and the value enumerator of f faithfully reflect each other. We then invoke the MacWilliams identity from coding theory, to obtain formulas for the number of binary zeros

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