

4. Weight enumerators of finite scalar product modules

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The associated Witt class is

$$w(\mathbf{E}_6) = \langle 1 \rangle \quad \text{in} \quad W(\mathbf{F}_3) .$$

CASE $R = \mathbf{E}_7$.

The definition is

$$\mathbf{ZE}_7 = \{ \sum_{i=1}^8 x_i e_i : 2x_i \in \mathbf{Z}, x_i - x_j \in \mathbf{Z}, \sum_{i=1}^8 x_i = 0 \} .$$

Here,

$$(\mathbf{ZE}_7)^\# = \mathbf{ZE}_7 \sqcup (\mathbf{ZE}_7 + z_1) ,$$

where

$$z_1 = \frac{1}{4} (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3(e_7 + e_8))$$

satisfies $(z_1, z_1) = \frac{3}{2}$ and is of minimal scalar square in its class *mod* \mathbf{ZE}_7 .

Again, z_1 is noted $x_1(\mathbf{E}_7)$ if convenient.

The Witt class $w(\mathbf{E}_7)$ is the generator $\langle 1 \rangle$ of $W(\mathbf{F}_2) = \mathbf{Z}/2\mathbf{Z}$.

CASE $R = \mathbf{E}_8$.

Here, $T(\mathbf{E}_8) = 0$. The associated Witt class is 0.

4. WEIGHT ENUMERATORS OF FINITE SCALAR PRODUCT MODULES

Let T be a finite abelian group with a non-degenerate bilinear form $b : T \times T \rightarrow \mathbf{Q}/\mathbf{Z}$.

Suppose that we have a decomposition of T as an orthogonal direct sum of subgroups T_1, \dots, T_s :

$$T = T_1 \boxplus T_2 \boxplus \dots \boxplus T_s .$$

Then we can define the weight $x^{w(u)} \in \mathbf{Z}[x_1, \dots, x_s]$ of an element $u \in T$ by tabulating its non-zero components in the decomposition $u = u_1 + u_2 + \dots + u_s$, $u_i \in T_i$, as

$$x^{w(u)} = x_1^{w(u_1)} \cdot x_2^{w(u_2)} \cdot \dots \cdot x_s^{w(u_s)} ,$$

where

$$w(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{if } u_i \neq 0. \end{cases}$$

If M is a subset of T , the *weight enumerator* of M is the polynomial

$$P_M(x_1, \dots, x_s) = \sum_{u \in M} x^{w(u)}.$$

We denote by q_i , $i = 1, \dots, s$ the order of the subgroup T_i .

We show in this section that MacWilliams duality is still valid in this more general setting:

THEOREM. *Let $M \subset T$ be a subgroup of the scalar product module $T = T_1 \oplus T_2 \oplus \dots \oplus T_s$. Set $q_i = \text{Card}(T_i)$, and let M^\perp be the subgroup orthogonal to M . Then, we have the formula, where $|M| = \text{Card}(M)$:*

$$P_{M^\perp}(x_1, \dots, x_s) = \frac{1}{|M|} \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot P_M\left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_s}{1 + (q_s - 1)x_s}\right).$$

Note that if some of the subgroups T_1, \dots, T_s are mutually isomorphic (or more generally have the same order), then we can write the decomposition of T in the form

$$T = n_1 T_1 \oplus n_2 T_2 \oplus \dots \oplus n_r T_r,$$

where $n_i T_i$ stands for the orthogonal sum

$$n_i T_i = T_i \oplus T_i \oplus \dots \oplus T_i$$

of n_i copies of T_i .

The weight of an element

$$u = (u_{1,1} + \dots + u_{1,n_1}) + \dots + (u_{r,1} + \dots + u_{r,n_r})$$

is then defined as

$$x^{w(u)} = x_1^{v_1} \cdot x_2^{v_2} \cdot \dots \cdot x_r^{v_r},$$

where v_i is the number of non-zero components of $u_{i,1} + \dots + u_{i,n_i}$ in $n_i T_i$.

The duality theorem then takes the seemingly more general form

$$P_{M^\perp}(x_1, \dots, x_r) = \frac{1}{\text{Card}(M)} \prod_{i=1}^r (1 + (q_i - 1)x_i)^{n_i} \cdot P_M\left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_r}{1 + (q_r - 1)x_r}\right).$$

This identity can be viewed as a system of linear equations for the coefficients of the weight enumerator polynomial P_M of any putative metabolizer $M = M^\perp$. If M exists, this system must be solvable in non-negative integers.

Proof of the duality theorem. One of the classical proofs of MacWilliams duality in a vector space over a finite field goes over with only insignificant changes. We repeat the argument for the reader's convenience.

Let $\chi: \mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{C}^*$ be the character given by $\chi(\alpha) = e^{2\pi i\alpha}$. Set $\beta(u, v) = \chi(b(u, v))$.

We cook up the function $f: T \rightarrow \mathbf{C}[x_1, \dots, x_s]$ given by

$$f(u) = \sum_{v \in T} \beta(u, v) \cdot x^{w(v)}$$

and evaluate $\sum_{u \in M} f(u)$ in two different ways, using the following lemma:

LEMMA.

$$\sum_{u \in M} \beta(u, v) = \begin{cases} \text{Card}(M) & \text{if } v \in M^\perp, \\ 0 & \text{if } v \notin M^\perp. \end{cases}$$

We first recall the proof of the lemma.

If $v \in M^\perp$, then $\beta(u, v) = 1$ for every $u \in M$, thus $\sum_{u \in M} \beta(u, v) = \text{Card}(M)$ as stated in this case.

If $v \notin M^\perp$, there is an element $u_1 \in M$ such that $b(u_1, v) \neq 0$, and then $\beta(u_1, v) \neq 1$. We have

$$\begin{aligned} \sum_{u \in M} \beta(u, v) &= \sum_{u \in M} \beta(u_1 + u, v) \\ &= \sum_{u \in M} \beta(u_1, v) \beta(u, v) = \beta(u_1, v) \sum_{u \in M} \beta(u, v). \end{aligned}$$

This implies the statement of the lemma for $v \notin M^\perp$.

We now proceed to the proof of the duality theorem.

Firstly,

$$\begin{aligned} \sum_{u \in M} f(u) &= \sum_{u \in M} \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} = \sum_{v \in T} \left(\sum_{u \in M} \beta(u, v) \right) \cdot x^{w(v)} \\ &= \sum_{v \in M^\perp} \text{Card}(M) \cdot x^{w(v)} = \text{Card}(M) \cdot P_{M^\perp}(x_1, \dots, x_s). \end{aligned}$$

Secondly,

$$\begin{aligned} f(u) &= \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} \\ &= \sum_{v_1 \in T_1, \dots, v_s \in T_s} \beta(u_1, v_1) \cdot \dots \cdot \beta(u_s, v_s) \cdot x_1^{w(v_1)} \cdot \dots \cdot x_s^{w(v_s)} \\ &= \prod_{i=1}^s \left(\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)} \right), \end{aligned}$$

where $u = u_1 + \dots + u_s$ is the decomposition of $u \in T = T_1 \boxplus \dots \boxplus T_s$.

Using the lemma again, we have

$$\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)} = \begin{cases} 1 + (q_i - 1)x_i & \text{if } u_i = 0, \\ 1 - x_i & \text{if } u_i \neq 0. \end{cases}$$

Thus,

$$f(u) = \prod_{i \in S} (1 + (q_i - 1)x_i) \cdot \prod_{i \in S'} (1 - x_i),$$

where $S \subset \{1, \dots, s\}$ is the set of indices i for which $u_i = 0$, and $S' \subset \{1, \dots, s\}$ the set of indices i for which $u_i \neq 0$.

Another way of writing $f(u)$ is

$$f(u) = \prod_{i=1}^s (1 - x_i)^{w(u_i)} \cdot (1 + (q_i - 1)x_i)^{1 - w(u_i)}.$$

Plugging this formula into $\sum_{u \in M} f(u)$, we get

$$\begin{aligned} \sum_{u \in M} f(u) &= \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot \sum_{u \in M} \prod_{i=1}^s \left(\frac{1 - x_i}{1 + (q_i - 1)x_i} \right)^{w(u_i)} \\ &= \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot P_M \left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_s}{1 + (q_s - 1)x_s} \right). \end{aligned}$$

Comparing the two expressions for $\sum_{u \in M} f(u)$, we get the theorem.

5. THE DEFICIENCY

The main further necessary condition for a root system to be contained in an even unimodular lattice of the same rank is provided by the notion of deficiency (Defekt) introduced and studied in [KV].

If R is a root system of rank n , the *deficiency* of R , denoted $d(R)$, is the difference $n - m$, where m is the maximal cardinality of a set $\{a_1, \dots, a_m\} \subset R$ of mutually orthogonal roots

$$(a_i, a_j) = 2\delta_{ij}, \quad \text{for all } 1 \leq i, j \leq m.$$

We use this notion only if all roots in R have the same scalar square 2.

If $R = R_1 \boxplus R_2$, then $d(R) = d(R_1) + d(R_2)$. The values of the deficiency for the irreducible root systems are

$$\begin{aligned} d(\mathbf{A}_l) &= \left[\frac{l}{2} \right], \\ d(\mathbf{D}_l) &= \begin{cases} 0 & \text{for } l \text{ even,} \\ 1 & \text{for } l \text{ odd,} \end{cases} \end{aligned}$$

$$d(\mathbf{E}_6) = 2, \quad d(\mathbf{E}_7) = d(\mathbf{E}_8) = 0.$$