3. The Brauer representation

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The algebra of (iii) contains the f_i's by definition. That (iii) implies (i) will follow if we can show that any element of $\cup_{t<n} \overrightarrow{\mathscr{A}(n, n; t)}$ is expressible as a product of F_i's. That this is true for diagrams having a straight through-string is a well known fact about the Temperley-Lieb algebra. But if D is an oriented diagram with less than n through-strings, either D has zero through-string and we are in the Temperley-Lieb situation, or $D \circ u^k$ has a straight through-string for some even k. Thus Du^k is a word on the F_i's and it suffices to show that $F_i u^2$ is a word on the F_i's for all i. It follows from a picture that $F_i u^{-2} = F_i F_{i+1} \dots F_n F_1 F_2 \dots F_{i-2}$. □

Remark 2.9. We leave it to the reader to show that Lemma 2.8 is true without the →'s if n is odd.

Remark 2.10. It follows from 2.8 that the elements v_t are in the algebra generated by the F_i's for $t < n$. We record the expression

$$v_{n-2}^2 = F_n \circ F_1 \circ F_2 \circ \cdots \circ F_n .$$

Thus rotations are unavoidable even if one is only interested in the structure of the algebra generated by the F_i's.

3. The Brauer representation

So far we have begged the important question of when the algebra $A(n, \delta)$ is semisimple. We do not have a complete answer for this but we shall show that it is semisimple whenever δ is an integer $\geqslant 3$, (and that $A(n, -2)$ is not semisimple for $n \geqslant 3$) by using a representation onto a C^*-algebra which we will show to be faithful for such δ. That the representation is faithful for n fixed and large integral (hence any large) δ is rather easy.

Definition 3.1. Let V be a vector space of dimension k and basis $w_1, w_2, \dots, w_k$. If the diagram $D \in D(n, n)$ has n connecting edges called ε, define $\beta(D) \in \mathrm{End}(\otimes^n V)$ by the matrix (with respect to the basis $\{w_{a_1} \otimes w_{a_2} \otimes \cdots \otimes w_{a_n} \mid a_i = 1, 2, \dots k\}$ of $\otimes^n V$)

$$\beta(D)_{a_1 a_2 \dots a_n}^{a_{n+1} \dots a_{2n}} = \prod_{\varepsilon} \delta(a_{s(\varepsilon)}, a_{b(\varepsilon)})$$

where $s(\varepsilon)$, $b(\varepsilon)$ are the two ends of the edge ε, labelled from 1 to $2n$, and, just in this formula, δ is the Kronecker δ.

LEMMA 3.2. *$D \mapsto \beta(D)$ defines a homomorphism of $B(n, k)$ (hence $A(n, k)$) onto a C^*-subalgebra of* $\mathrm{End}(\otimes^n V)$.

Proof. This is just the orthogonal case of [B]. The C^*-structure is that for which V is a Hilbert space with orthonormal basis $\{w_i\}$, and it is clear that the adjoint of D is just D read backwards.

Remark 3.3. Since finite-dimensional C^*-algebras are semisimple, this proves that $\beta(B(n, k))$ is always semisimple. Further note that $\beta(A(n, k))$ is also a C^*-algebra.

THEOREM 3.4. *For $k \geqslant 2$, β restricted to $TL(n, k)$ is faithful for all n.*

Proof. The normalized trace on $\mathrm{End}(\otimes^n V)$ defines a Markov trace on $TL(n, k)$ with Markov parameter k^2. Thus by the calculation of [J] or [GHJ], the structure of $\beta(TL(n, k))$ is known and it has the same dimension as $TL(n, k)$.

THEOREM 3.5. *For $k \geqslant 3$, β restricted to $A(n, k)$ is faithful for all n.*

Proof. Let $x = \sum_{D \in \mathscr{A}(n,n)} \lambda_D D$ ($\lambda_D \in \mathbf{C}$) be such that $\beta(x) = 0$. We have seen that $\mathscr{A}(n, n; 0)$ actually consists of planar diagrams so by 3.4 we may suppose that $\lambda_D \neq 0$ for some $D \in \mathscr{A}(n, n; t)$, $t \geqslant 1$. Thus by pre- and post-multiplying x by suitable powers of u, we may assume $\lambda_D \neq 0$ for some D with a straight line joining the inner and outer $*$'s. Now split V as $\mathbf{C}w_1 \oplus w_1^\perp$. Since dim $V > 2$, dim $w_1^\perp \geqslant 2$. Let P be orthogonal projection from $\otimes^n V$ onto $w_1 \otimes (\otimes^{n-1} w_1^\perp)$. If D is a diagram with the inner and outer $*$'s not connected, $P\beta(D)P = 0$. Also, the set of diagrams with a straight line between the $*$'s is in obvious bijection with $\mathscr{P}(n-1, n-1)$. Thus $0 = P\beta(x)P = \sum_{D \in \mathscr{P}(n-1,n-1)} \lambda_D P\beta(D)P$ and not all the λ_D's are zero.

But the matrix of $P\beta(D)P$ with respect to the basis

$$\{w_1 \otimes (w_{a_1} \otimes \cdots \otimes w_{a_{n-1}}) \mid a_i = 2, 3, \ldots, k\}$$

is clearly that of "$\beta(D)$" for parameters $k-1$ and $n-1$. By 3.4 we conclude $\sum_{D \in \mathscr{P}(n-1,n-1)} \lambda_D D = 0$, a contradiction. □

COROLLARY 3.6. *$\mathscr{A}(n, k)$ and $\overrightarrow{\mathscr{A}(n, k)}$ are semisimple fo k an integer $\geqslant 3$.*

The question naturally arises of finding those values of δ and n for which $\mathscr{A}(n, \delta)$ is semisimple. We observe that for $\delta = -2$, the algebra $\mathscr{A}(n, \delta)$ is not semisimple for $n > 2$. This is because we may use the Brauer representation corresponding to the symplectic case.

Then $\beta(f_1)$ is represented on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$

$$\text{by the matrix } \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}$$

using a symplectic basis of $\mathbf{C}^2$, and $\beta(u)$ is the obvious cyclic permutation on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$. But then $2 - \beta(f_1)$ is the transposition on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ exchanging the first two copies of $\mathbf{C}^2$. Thus the image of $\mathscr{A}(n, \delta)$ is the same as that of the group algebra of the symmetric group.

4. The cylindrical trace

There is a natural trace functional tr on $A(n, \delta)$ defined by $\mathrm{tr}(D) = \delta^{n(D)}$, $n(D)$ being the number of closed loops formed on the cylinder if the inside and outside boundaries of the annulus are identified. We will call this trace the cylindrical trace.

Note 4.1. This trace exists in fact on the whole Brauer algebra — it could be defined in terms of partitions as $\mathrm{tr}(D) = \delta^{n(D)}$ where $n(D)$ is the number of equivalence classes for the equivalence relation generated by D itself and the relation which identifies each point on the top with the corresponding point on the bottom.

Note 4.2. One has the relation $n(D_1 \circ D_2) = n(D_2 \circ D_1)$ so one might try to define a more general trace by replacing δ by an arbitrary complex number. But $n(\alpha, \beta) \neq n(\beta, \alpha)$ in general so one is forced to choose δ.

If δ is a value for which $A(n, \delta)$ is semisimple we know that $A(n, \delta)$ is a direct sum of matrix algebras, so our cylindrical trace is determined by its value on a minimal idempotent in each matrix algebra summand. We will calculate these "weights" of the trace. In order to do this we will need detailed information on the multiplicities of u in each irreducible representation of $A(n,\delta)$.

Definition 4.3. For $n \geqslant t > 0$ the group $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/t\mathbf{Z} (= \{(a, b) \mid a = 0, \ldots, n-1; b = 0, \ldots t-1\}$ acts by linear transformations on $\mathscr{A}(t, n; t)$ by $(a, b)(D) = u^a \circ D \circ u^b$. (The u's on the left and right in this formula are of course different if $n \neq t$.) Let $F_{n,t}(a, b)$ be the number of fixed points for (a, b). Let $F_n(a)$ be the number of fixed points for the action $D \mapsto D \circ u^a$ of $a \in \mathbf{Z}/n\mathbf{Z}$ on $\mathscr{A}(n, 0)$.