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Thus, as  $i \rightarrow \infty$ , we see that  $x'_i, y'_i, z'_i$  converge to the same point, say  $z'$ , on  $[x, y]$ . Thus  $d(x_i, y'_i) + d(y_i, x'_i) - d(x_i, y_i)$  converges to zero. Since  $d(x_i, z_i) + d(y_i, z_i) - d(x_i, y_i)$  also converges to zero, we have that  $d(y'_i, z_i) + d(z_i, x'_i)$  converges to zero. Since  $d(z_i, z'_i) \leq d(z'_i, y'_i) + d(y'_i, z_i) \leq 4\delta_i + d(y'_i, z_i)$  we see that the  $z_i$  converge to the point  $z'$  on our original geodesic segment  $[x, y]$ . Thus  $z$ , the midpoint of our arbitrary geodesic from  $x$  to  $y$ , coincides with the midpoint of our fixed geodesic. Repeating the argument we see that these geodesics must agree at a dense set of points, and hence everywhere. Since geodesic triangles in  $C_i$  are  $\delta_i$ -slim, and geodesics in  $C$  all arise as limits of geodesics in  $C_i$ , we see that geodesic triangles in  $C$  must be 0-slim, and hence  $C$  is an **R**-tree.  $\square$

*Remark.* If one has a sequence of  $\delta_i$ -hyperbolic spaces  $C_i$ , with  $C_i \rightarrow C$  and  $\delta_i \rightarrow \delta > 0$ , then one can extend the preceding argument to show that  $C$  is  $\delta'$ -hyperbolic (with  $\delta' = 19\delta$ , for example).

## SECTION 2: THE PROOF OF PAULIN'S THEOREM

In this section we shall prove the following theorem of F. Paulin [P4].

**2.1 THEOREM (Paulin).** *If  $\Gamma$  is a word hyperbolic group and  $\text{Out}(\Gamma)$  is infinite, then  $\Gamma$  acts by isometries on an **R**-tree with virtually cyclic segment stabilizers and no global fixed points.*

In its outline, the proof given below is very similar to Paulin's original proof, except that we use Hausdorff-Gromov convergence instead of the equivariant Gromov convergence used by Paulin. In particular, this allows us to avoid the difficulties discussed in the next section.

Let  $S$  be a finite set of generators for  $\Gamma$  and let  $X = X(\Gamma, S)$  denote the Cayley graph of  $\Gamma$  with respect to  $S$ , as defined in the introduction.  $\Gamma$  is the vertex set of  $X$  and receives the induced metric. The hypothesis that  $\Gamma$  is word hyperbolic means precisely that there exists  $\delta > 0$  such that  $X$  is a  $\delta$ -hyperbolic geodesic metric space. Note that with our definition of a Cayley graph, the endpoints of each edge are distinct, and there is at most one edge joining each pair of vertices; hence the action of  $\Gamma$  on itself by left multiplication can be extended linearly across edges in a unique way to give an isometric action of  $\Gamma$  on  $X$ .

The proof of Theorem 2.1 will be broken into a number of smaller results. We begin by noting that, because  $\text{Out}(\Gamma)$  is infinite, we can choose a sequence

of automorphisms  $\{\phi_i\}_{i \in \mathbb{N}}$  such that none of the  $\phi_i$  is an inner automorphism and no two of the  $\phi_i$  have the same image in  $\text{Out}(\Gamma)$ . For each  $i \in \mathbb{N}$  we consider the function  $f_i : X \rightarrow [0, \infty)$  defined by:

$$(2.2) \quad f_i(x) = \max_{s \in S} d(x, \phi_i(s)x) .$$

This function has been used by Bestvina in his study of degeneration of real hyperbolic structures [B], and our use of this function is similar to his. (A similar idea was used earlier in a different context by Thurston [T, Prop. 1.1].)

Note that  $f_i$  takes on integer values at vertices and midpoints of edges in  $X$ , and its restriction to half-edges is linear. It follows that  $f_i$  attains its infimum (which is an integer) at some point,  $x_i \in X$  say. (In the case where  $\Gamma$  is not virtually cyclic one can also see this by showing that  $f_i$  is a proper map, i.e., a map with the property that the inverse image of a compact set is compact.)

Let

$$(2.3) \quad \begin{aligned} \lambda_i &= \max_{s \in S} d(x_i, \phi_i(s)x_i) \\ &= \inf_{x \in X} \max_{s \in S} d(x, \phi_i(s)x) . \end{aligned}$$

We fix a definite choice of points  $x_i$  with the above property.

For future reference, we note that by passing to a subsequence of the  $\phi_i$  we may assume there is a single element  $s_0 \in S$  such that  $\lambda_i = d(x_i, \phi_i(s_0)x_i)$  for all  $i \in \mathbb{N}$ . We also note that with the above choice of  $x_i$ , the triangle inequality yields:

$$(2.4) \quad d(x_i, \phi_i(\gamma)x_i) \leq \lambda_i d(e, \gamma) .$$

Following Paulin, we next note that because  $\text{Out}(\Gamma)$  is infinite, the sequence  $\lambda_i$  must be unbounded. For suppose that there were a uniform bound,  $\rho$  say, on the value of  $\lambda_i$ . Then for any vertex  $y_i \in X$  closest to  $x_i$ , we would have  $d(e, y_i^{-1}\phi_i(s)y_i) = d(y_i, \phi_i(s)y_i) \leq \rho + 2$  for all  $s \in S, i \in \mathbb{N}$ . But there are only finitely many vertices in the ball of radius  $\rho + 2$  about  $e$ , so this bound would imply the existence of integers  $n \neq m$  such that  $y_n^{-1}\phi_n(s)y_n = y_m^{-1}\phi_m(s)y_m$  for all  $s \in S$ . Whence  $\phi_n$  and  $\phi_m$  would be equal in  $\text{Out}(\Gamma)$ , contrary to hypothesis. Thus we have shown that the sequence of numbers  $\{\lambda_i\}_{i \in \mathbb{N}}$  is unbounded, so we may pass to a subsequence  $\{\lambda_n\}_{n \in \mathbb{N}}$  which is *strictly increasing* and assume that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Consider the sequence of metric spaces  $X_k = (X, d_k)$ , where  $d_k := d/\lambda_k$  is the original metric on  $X$  scaled down by  $\lambda_k$ . In what follows we shall intermittently use both the original metric  $d$  and the scaled metric  $d_k$ , specifying which on each occasion and, where appropriate, using the formal notation  $(Y, d)$  for a metric space which consists of the set  $Y$  together with a distance function  $d$ . But for the moment, the most important distinction between the  $X_k$  will be that we shall regard  $\Gamma$  as acting on  $X_k$  via  $\phi_k$ , and think of our chosen point  $x_k$ , at which the minimax  $\lambda_k$  is attained, as a *basepoint* in  $X_k$ . More precisely, we consider the sequence of pointed  $\Gamma$ -spaces  $(X_k, x_k)$ , where the action of  $\gamma \in \Gamma$  on  $X_k$  is  $x \mapsto \phi_k(\gamma)x$ .

We wish to use the hyperbolic nature of  $X_k$  to approximate it by a sequence of star-like compact subsets  $X_k(P_i)$  centred at  $x_k$ . To this end, we fix a sequence of finite subsets  $\{e\} = P_0 \subseteq P_1 \subseteq P_2 \cdots \subseteq P_i \subseteq \cdots$  which exhaust  $\Gamma$ . Let  $n_i = |P_i|$  denote the cardinality of  $P_i$ . The desired subsets of  $X_k$  are defined inductively as follows:  $X_k(P_0) = \{x_k\}$ , and  $X_k(P_i)$  is the union of  $n_i - 1$  geodesic segments, those in  $X_k(P_{i-1})$  together with a choice of geodesic segment from  $x_k$  to each element of  $\{\phi_k(\gamma)x_k \mid \gamma \in P_i - P_{i-1}\}$ .

We next ‘fatten-up’ each of the sets  $X_k(P_i)$  by taking its closed  $\delta$ -neighbourhood in the metric  $d$ . Henceforth we shall denote this neighbourhood  $V_k^i$ . Let  $d_{i,k}$  be the induced *path metric* on  $V_k^i$ . As we discussed in Section 1,  $(V_k^i, d_{i,k})$  is a geodesic metric space. It is also important to notice that the induced path metric which  $V_k^i$  receives from  $d_k$  is  $d_{i,k}/\lambda_k$ . The following lemma is suggested by an argument of B. Bowditch [Bo].

2.5 LEMMA. *With the above notation, for all  $x, y \in V_k^i$  we have:*

$$d(x, y) \leq d_{i,k}(x, y) \leq d(x, y) + 4\delta.$$

*Proof.* The left-most inequality comes from the general fact that for any subspace of a geodesic metric space the induced metric is dominated by the induced path metric. In order to establish the other inequality, we first note that  $X_k(P_i)$  is  $\delta$ -convex in  $(X_k, d)$ , in the sense that if a geodesic segment in  $X_k$  joins a pair of points  $x, y \in X_k(P_i)$ , then this geodesic segment lies entirely within the closed  $\delta$ -neighbourhood  $V_k^i$  of  $X_k(P_i)$ .

Given  $x, y \in V_k^i$ , we fix points  $z, w \in X_k(P_i)$  closest to  $x$  and  $y$  respectively. (Such points are not unique in general.) Let  $[x, z]$ ,  $[z, w]$  and  $[w, y]$  be choices of geodesic segments joining  $x$  to  $z$ ,  $z$  to  $w$  and  $w$  to  $y$ , respectively. Each is contained in  $V_k^i$ , and hence so is the broken geodesic  $[x, z, w, y]$  obtained by concatenating them. The length of this broken geodesic is at most  $d(z, w) + 2\delta \leq d(x, y) + 4\delta$ . Hence  $d_{i,k}(x, y) \leq d(x, y) + 4\delta$ .  $\square$



The subspace  $V_k^i$  forms a good substitute for the notion of a convex hull for  $\phi_k(P_i)x_i$  in  $X_k$ . According to the above lemma, geodesics in  $(V_k^i, d_{i,k})$  are  $(1, 4\delta)$ -quasigeodesics in  $(X_k, d)$ , and hence by [GH, p. 82] there exists a constant  $\eta = \eta(\delta)$  (independent of  $k, i$ ) such that geodesic triangles in  $(V_k^i, d_{i,k})$  are  $\eta$ -slim. Thus we have proved the first part of:

**2.6 LEMMA.** *There exists a constant  $\eta = \eta(\delta)$  such that, for all  $k \in \mathbb{N}$ , with respect to the path metric  $d_{i,k}$  on  $V_k^i$ , geodesic triangles in  $V_k^i$  are  $\eta$ -slim. Moreover, for fixed  $i$ , with respect to the (scaled) path metrics  $d_{i,k}/\lambda_k$ , the metric spaces  $\{V_k^i\}_{k \in \mathbb{N}}$  are uniformly compact.*

*Proof.* It remains to prove the assertion of the second sentence. We follow an argument of Bestvina [B]. Until further notice we work with the metric  $d$ . Let  $\mu_i$  be the maximum of the integers  $\{d(e, \gamma) \mid \gamma \in P_i\}$ . Each of the geodesic segments used to define  $X_k(P_i)$  has length at most  $\mu_i \lambda_k$  (by (2.4)). Therefore, given  $\varepsilon > 0$ , we can cover  $X_k(P_i)$  by  $2n_i \mu_i / \varepsilon$  segments of length at most  $\lambda_k \varepsilon / 2$ . (Recall that  $n_i = |P_i|$ .) Hence, if  $\lambda_k \varepsilon > 2\delta$ , then in order to cover  $V_k^i$  we need at most  $2n_i \mu_i / \varepsilon$  balls of radius  $\lambda_k \varepsilon$ . But we arranged that  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ , so this is true for large  $k$ .

Now we change viewpoints and work with the scaled metric  $d_k$  on  $X_k$ , and the induced path metric on  $V_k^i$ . In this setting, the preceding argument shows that for large  $k$  one needs only  $2n_i \mu_i / \varepsilon$  balls of radius  $\varepsilon$  to cover  $V_k^i$ . Since the path metric on  $V_k^i$  and the restriction to  $V_k^i$  of  $d_k$  differ by at most an additive constant of  $4\delta/\lambda_k$ , we have thus established the existence of a uniform  $\varepsilon$ -count for the  $\{V_k^i\}_{k \in \mathbb{N}}$  both when equipped with the restriction of the metrics  $d_k$  and when equipped with the induced path metrics. Because they are *path* metric spaces, a uniform  $\varepsilon$ -count also yields a bound on the diameter of the  $V_k^i$ .  $\square$

Continuing with the proof of Paulin's theorem, we fix an integer  $j$  and suppose that we are given a positive constant  $\varepsilon$ . According to the preceding lemma, we can choose  $\varepsilon$ -nets  $N_\varepsilon(k, j)$  for  $V_k^j$  on whose cardinalities there is a bound independent of  $k$ . We may also assume that the set  $N_\varepsilon(k, j)$  includes  $\phi_k(P_j)x_k$ . Since, for fixed  $j$ , the  $N_\varepsilon(k, j)$  are finite metric spaces of uniformly bounded cardinality and diameter, we can pass to a subsequence (using a diagonal type argument, as in Section 1) so as to assume that, for all  $\gamma, \gamma' \in P_j$ , the sequence of numbers  $d_{j,k}(\phi_k(\gamma)x_k, \phi_k(\gamma')x_k)$  converges as  $k \rightarrow \infty$ . Passing to a further subsequence which is convergent in the Hausdorff-Gromov topology we obtain a limit metric space  $L_{\varepsilon,j}$  (whose cardinality will be no greater than that of the  $N_\varepsilon(k, j)$ ). As a basepoint in the

limit space we choose the limit of the sequence  $x_k$ , and we christen this point  $x_\infty$ . For each  $\gamma \in P_j$ , we denote the limit of the sequence  $\phi_k(\gamma)x_k$  by  $\gamma x_\infty$ .

We next take an  $\varepsilon/2$ -net for  $V_k^j$  which is constructed so as to include the previously chosen  $\varepsilon$ -net. Passing to a subsequence if necessary, we obtain a finite limit metric space  $L_{\varepsilon/2,j}$ . We proceed in this manner, taking finer  $\varepsilon$ -nets, and at each stage including the previous (coarser) ones and extracting convergent subsequences to obtain finite limit metric spaces. The natural inclusions of each  $\varepsilon$ -net into its refinements gives a natural identification of points in the limit, so it is not too abusive a notation to write:

$$L_{\varepsilon,j} \subset L_{\varepsilon/2,j} \cdots \subset L_{\varepsilon/2^n,j} \subset \cdots$$

We define  $L_j$  to be the direct limit of this sequence, that is,  $L_j = \bigcup \{L_{\varepsilon/2^n,j} \mid n \in \mathbf{N}\}$ . We denote by  $\hat{L}_j$  the metric completion of  $L_j$ . Since the diameters of the  $V_k^j$  are uniformly bounded in the scaled metrics, we see that  $\hat{L}_j$  is a complete space of finite diameter, and hence is compact.

By choosing a diagonal type subsequence and renumbering, we obtain the following array of spaces with convergence in both the horizontal and vertical directions:

$$\begin{array}{ccccccc} N_\varepsilon(1,j) & \subseteq & N_{\varepsilon/2}(2,j) & \subseteq & \cdots & \subseteq & N_{\varepsilon/2^n}(1,j) & \subseteq & \cdots & \subseteq & V_1^j & \subseteq & X_1 \\ N_\varepsilon(2,j) & \subseteq & N_{\varepsilon/2}(2,j) & \subseteq & \cdots & \subseteq & N_{\varepsilon/2^n}(2,j) & \subseteq & \cdots & \subseteq & V_2^j & \subseteq & X_2 \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \\ N_\varepsilon(m,j) & \subseteq & N_{\varepsilon/2}(m,j) & \subseteq & \cdots & \subseteq & N_{\varepsilon/2^n}(m,j) & \subseteq & \cdots & \subseteq & V_m^j & \subseteq & X_m \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \\ L_{\varepsilon,j} & \subseteq & L_{\varepsilon/2,j} & \subseteq & \cdots & \subseteq & L_{\varepsilon/2^n,j} & \subseteq & \cdots & \subseteq & \hat{L}_j & & \end{array}$$

Our next goal is to show that as  $k \rightarrow \infty$  the  $V_k^j$  actually converge to  $\hat{L}_j$  in the Hausdorff-Gromov topology. We have that  $N_{\varepsilon/2^n}(m,j)$  is  $\varepsilon/2^{n-1}$  close to  $V_m^j$  for all  $m$ . After passing to yet another diagonal type subsequence, we may assume that  $N_{\varepsilon/2^n}(m,j)$  is  $\varepsilon/2^{m-1}$  close to  $L_{\varepsilon/2^n,j}$  for all  $m \geq n$ . Thus  $V_m^j$  and  $L_{\varepsilon/2^n,j}$  are  $\varepsilon/2^{n-2}$  close for  $m \geq n$ . On the other hand,  $L_{\varepsilon/2^n,j}$  and  $L_{\varepsilon/2^{n+1},j}$  are  $\varepsilon/2^{n+1}$  close (since any choice of  $\varepsilon/2^n$  and  $\varepsilon/2^{n+1}$  nets of  $V_k^j$  are  $\varepsilon/2^{n+1}$  close). Thus  $L_{\varepsilon/2^n,j}$  is  $\sum_{i \geq n} \varepsilon/2^i$  close to  $L_j$  and  $\hat{L}_j$ . Hence  $V_n^j$  and  $\hat{L}_j$  are  $\varepsilon/2^{n-3}$  close, so  $V_n^j$  converges to  $\hat{L}_j$ , in the Hausdorff-Gromov topology, as  $n \rightarrow \infty$ .

Notice that, by (1.9) and (2.6), the spaces  $\hat{L}_j$  are  $\mathbf{R}$ -trees of finite diameter, because  $V_k^j$  is  $\eta/\lambda_k$ -hyperbolic and  $\lambda_k \rightarrow \infty$ . It is also useful to observe that  $\hat{L}_j$  is spanned by  $\gamma x_\infty$ , with  $\gamma \in P_j$ . Furthermore, the  $X_k(P_j)$  themselves converge to  $\hat{L}_j$  because  $X_k(P_j)$  and  $V_k^j$  are  $4\delta/\lambda_k$ -close and  $\lambda_k \rightarrow \infty$ . However, in what follows it is most convenient to still work with  $V_k^j$  rather than  $X_k(P_j)$  when we need to take a choice of geodesic between two points of  $X_k(P_j)$ . Also, because the scaled path metric on  $V_k^j$  and the induced metric  $d_k/\lambda_k$  differ only by  $4\delta/\lambda_k$ , which tends to 0 as  $k \rightarrow \infty$ , henceforth it is not important to keep track of the difference between these two metrics.

By construction, all of our  $\varepsilon/2^n$ -nets include the set  $\{\phi(\gamma)x_k \mid \gamma \in P_j\}$  and each of the sequences  $d_k(\phi(\gamma)x_k, \phi(\gamma')x_k)$  converges. Thus, if we denote by  $x_\infty \in \hat{L}_j$  the 'limit' of the  $x_k$ , and by  $\gamma x_\infty$  the limit of the  $\phi(\gamma)x_k$ , then we see that  $d(\gamma x_\infty, \gamma' x_\infty)$  (distance in  $\hat{L}_j$ ) is independent of  $j$ . Since the tree  $\hat{L}_j$  is the convex hull of the points  $\gamma x_\infty$ , we can define an isometric embedding of  $\hat{L}_j$  into  $\hat{L}_{j+1}$  for all  $j$  and hence obtain an  $\mathbf{R}$ -tree by taking the direct limit of the resulting system of inclusions. We denote the direct limit metric space with basepoint (which as the limit of  $\mathbf{R}$ -trees is itself an  $\mathbf{R}$ -tree) by  $(X_\infty; x_\infty)$ . The final important observation to make is that  $\Gamma$  acts isometrically on  $X_\infty$ , because it acts isometrically on the subset  $\{\gamma x_\infty\}_{\gamma \in \Gamma}$  (by left translation), and the convex hull of this subset is the whole of  $X_\infty$ .

Let us now examine the nature of the action of  $\Gamma$  on  $X_\infty$ . We claim that it has the following properties:

- (1) There is no point of  $X_\infty$  whose stabilizer is the whole of  $\Gamma$ .
- (2) The stabilizer of every non-trivial segment in  $X_\infty$  is virtually cyclic.

To see that (1) is true, let us see what would happen if it were to fail. Suppose that  $\Gamma$  were to stabilize a point  $z_\infty \in X_\infty$ . We fix a segment  $z_\infty \in [\gamma x_\infty, \gamma' x_\infty] \subseteq \hat{L}_j$ . Up to the taking of subsequences, we have that the closures in  $\hat{L}_j$  of the images of the geodesic segments  $[\gamma x_k, \gamma' x_k] \subseteq V_k^j$  converge (in the Hausdorff metric) to  $[\gamma x_\infty, \gamma' x_\infty]$ , and we fix points  $z_k \in [\gamma x_k, \gamma' x_k]$  which converge to  $z_\infty$ . We then choose  $j$  large enough to ensure that  $S \subset P_j$  (recall that  $S$  is our fixed finite generating set for  $\Gamma$ ), and  $l$  large enough to ensure that  $P_j P_j \subset P_l$ .

We have, for every  $s \in S$ , geodesics  $[s\gamma x_k, s\gamma' x_k] := s \cdot [\gamma x_k, \gamma' x_k]$  in  $V_k^l$ , and (by definition of the action on  $X_\infty$ ) the closures of their images in  $\hat{L}_l \subseteq X_\infty$  converge to  $[s\gamma x_\infty, s\gamma' x_\infty]$ . Moreover,  $\{s z_k\}_{k \in \mathbf{N}}$  converges to  $s \cdot z_\infty = z_\infty$ , so for large  $k$  we have that  $d_k(s \cdot z_k, z_k) < 1/4$  in the *scaled*

metric of  $X_k$ . Hence  $d(s \cdot z_k, z_k) < \lambda_k/4$ , for large  $k$ , in the original metric on  $X_k$ . But this contradicts the definition of  $\lambda_k$ .

*Remark.* The preceding argument actually shows that for every finite set  $P \subseteq \Gamma$  which fixes  $z_\infty$ , given any  $\varepsilon > 0$  one has that for  $k$  sufficiently large  $z_k$  and  $\gamma z_k$  are  $\varepsilon$ -close, in the scaled metric  $d_k$ , for every  $\gamma \in P$ .

We next need to show that segment stabilizers are virtually cyclic. This seems to be the place where some sort of discreteness assumption on  $\Gamma$  is needed. In the classical real-hyperbolic case, Margulis' Lemma implies the result for discrete actions (see [B] and [P2]). Since we are using Cayley graphs and the group actions are (almost) free there is still some sort of discreteness and Paulin gives a delicate argument to show that segment stabilizers are virtually cyclic. The following algebraic lemma is taken from [P4]:

**2.7 LEMMA.** *Let  $G$  be a finitely generated group. If the set of commutators  $\{aba^{-1}b^{-1} \mid a, b \in G\}$  is finite, then  $G$  is virtually abelian.*

*Proof.* The action of  $G$  on itself by conjugation determines a map  $G \rightarrow \text{Aut}(\Gamma)$ , whose image is  $\text{Inn}(G)$  and whose kernel is the centre of  $G$ ; it suffices to prove that  $\text{Inn}(G)$  is finite. If  $A$  is a finite generating set for  $G$ , then the action of  $g \in G$  by conjugation is determined by its action on the elements  $a \in A$ . But  $g^{-1}ag = (g^{-1}aga^{-1})a$ , and by hypothesis there are only finitely many possibilities,  $M$  say, for the commutator  $g^{-1}aga^{-1}$ . Hence the cardinality of  $\text{Inn}(G)$  is at most  $M^{|A|}$ .  $\square$

We proceed with the proof of assertion (2) on segment stabilizers. We call a subgroup *large* if it contains a non-abelian free subgroup (for hyperbolic groups this is equivalent to not having a cyclic subgroup of finite index). Suppose that a large subgroup  $G$  of  $\Gamma$  stabilizes a non-trivial segment  $e \subseteq X_\infty$ . If  $e$  is finite, then a subgroup of index 2 in  $G$  fixes  $e$  pointwise. If  $e$  is infinite, a subgroup of index 2 in  $G$  acts as translations on a ray in  $e$  and thus a large subgroup of  $G$ , obtained by taking commutators, fixes a segment of positive length in  $e$  pointwise. Thus, in any case, if a large subgroup of  $\Gamma$  stabilizes a segment, then a (perhaps smaller) large subgroup of  $\Gamma$  fixes a segment  $e$  of positive length pointwise. Therefore, in order to complete the proof of Paulin's theorem, it suffices to show that if a subgroup of  $\Gamma$  fixes a segment of  $X_\infty$  pointwise, then that subgroup is virtually cyclic. Let  $D$  denote the length of such a segment which is fixed pointwise by the subgroup  $G \subset \Gamma$ , and let  $z$  and  $z'$  denote the endpoints of the segment.

We fix  $\varepsilon > 0$  small (to be estimated later) compared to  $D$ , and  $k$  so large that if  $z_k, z'_k \in X_k$  correspond to  $z, z' \in X$  then  $|d(z_k, z'_k) - D| < \varepsilon$ . We fix

a geodesic segment  $[z_k, z'_k]$  from  $z_k$  to  $z'_k$  in  $X_k$ . Given any finite subset  $P \subseteq G$ , we choose a finite subset  $Q \subseteq G$  which contains all products of length  $\leq 4$  in  $s, t, s^{-1}, t^{-1}$ , as  $s$  and  $t$  vary over  $P$ . We choose  $k$  large enough so that  $z_k, z'_k$  are moved by less than  $\varepsilon$  by each  $\gamma \in Q$  with respect to the scaled metric  $d_k = d/\lambda_k$ . If  $D > 3\varepsilon + (24\delta/\lambda_k)$  then if we omit segments of  $d$ -length  $\lambda_k\varepsilon + 12\delta$  from the ends of  $[z_k, z'_k]$ , the remaining sub-segment is non-empty; call this segment  $C_k$ . We assume that  $\varepsilon$  is small enough to satisfy the above inequality; we shall place further restrictions on  $\varepsilon$  later.

Now we use the original metric  $d$  on  $X_k$ . From the proof 'slim  $\Rightarrow$  thin' (see [Sho] p. 17), if  $x \in C_k$  then  $\gamma x$  is within  $12\delta$  of  $[z_k, z'_k]$ . We denote by  $\gamma_*x$  the projection of  $\gamma x$  on  $[z_k, z'_k]$ . Of course, the 'projection' is not uniquely defined, but the preceding sentence is true no matter which closest point on  $[z_k, z'_k]$  one chooses — we fix a definite choice for each  $x \in C_k$ , thus defining a map  $\gamma_*: C_k \rightarrow [z_k, z'_k]$  for each  $\gamma$ . Next, we omit segments of length  $5(\lambda_k\varepsilon + 12\delta)$  from the ends of  $[z_k, z'_k]$  and denote the remaining long segment by  $E_k \subseteq C_k$ . The map  $C_k \rightarrow [z_k, z'_k]$  just defined restricts to a map  $E_k \rightarrow [z_k, z'_k]$ ; we continue to denote this map by  $\gamma_*$ . Notice that this map is a  $24\delta$ -isometry, that is to say, it distorts distances by at most an additive constant of  $24\delta$ ; in fact it is  $24\delta$  close to a translation of  $E_k$  along  $[z_k, z'_k]$ . (Here, and in what follows, the terminology  $\eta$ -close is used to describe functions  $f, g$  with the same domain such that  $d(f(x), g(x)) < \eta$  for all points in their common domain.)

Note that on  $E_k$  the maps  $s_*, s_*t_*, s_*t_*(s^{-1})_*, s_*t_*(s^{-1})_*(t^{-1})_*$  etc. are well-defined and uniformly close to translations. Choose  $M = \text{Max}\{5(\lambda_k\varepsilon + 12\delta), 600\delta\}$ . We will denote by  $e_k$  the segment obtained from  $[z_k, z'_k]$  by omitting segments of length  $M$  from the ends. We have  $e_k \subset E_k$ . To make sure that  $e_k \neq \emptyset$  we assume  $D - \varepsilon > 5\varepsilon + (60\delta/\lambda_k)$ , we also assume  $D - \varepsilon > (600\delta/\lambda_k)$ . Since  $\lambda_k \rightarrow \infty$ , we can choose large enough  $k$  and small enough  $\varepsilon$  so that the above conditions are satisfied.

We shall consider the restrictions  $\gamma_*: e_k \rightarrow C_k$  to  $e_k$  of the maps  $\gamma_*$  defined above; we retain the notation  $\gamma_*$  for these restricted maps. Our goal is to obtain a bound (independent of  $|Q|$ ) on the number commutators  $tst^{-1}s^{-1}$  in  $Q$  by estimating how close the action of such a commutator on  $e_k$  is to the identity map. We first compare  $t_*s_*(t^{-1})_*(s^{-1})_*$  to  $tst^{-1}s^{-1}$ . Observe that, since the maps  $s$  and  $s_*$  are  $12\delta$  close,  $ts$  and  $t(s_*)$  are  $12\delta$  close (the left-action of  $\Gamma$  on  $X_k$  is by isometries in the metric  $d$ ). Hence,  $(ts)_*$  and  $t_*s_*$  are  $36\delta$  close. Comparing successively  $tst^{-1}s^{-1}$ ,  $(tst^{-1}s^{-1})_*$ ,  $(ts)_*(t^{-1}s^{-1})_*$ ,  $t_*s_*(t^{-1})_*(s^{-1})_*$  shows that  $tst^{-1}s^{-1}$  and  $t_*s_*(t^{-1})_*(s^{-1})_*$  are  $(12 + 36 + 108)\delta$  close.

Next, we compare  $t_*s_*(t^{-1})_*(s^{-1})_*$  to the identity map on  $e_k$ . Since  $s_*(t^{-1})_*$  and  $(t^{-1})_*s_*$  are  $72\delta$  close to the same translation, and translations commute, we have that  $t_*s_*(t^{-1})_*(s^{-1})_*$  and  $t_*(t^{-1})_*s_*(s^{-1})_*$  are  $(144 + 24)\delta$  close. Moreover,  $t_*(t^{-1})_*$  and  $s_*(s^{-1})_*$  are  $36\delta$  close to the identity. Thus  $t_*(t^{-1})_*s_*(s^{-1})_*$  is  $108\delta$  close to the identity. Hence  $t_*s_*(t^{-1})_*(s^{-1})_*$  is  $276\delta$  close to the identity. Combining this with the estimate in the previous paragraph we have that the restriction of  $(tst^{-1}s^{-1})$  to  $e_k$  is  $532\delta$  close to the identity on  $e_k$ . Therefore, a vertex close to the midpoint of  $e_k$  is moved by less than  $532\delta + 2$  by  $tst^{-1}s^{-1}$ . Thus  $tst^{-1}s^{-1}$  lies in the ball of radius  $532\delta + 2$  about the identity in  $\Gamma$ , and we have the desired bound on the number of commutators in the arbitrary finite subset  $P \subset G$ .

Now Lemma 2.7 implies that  $G$  is virtually abelian. But every abelian subgroup of a hyperbolic group is virtually cyclic. Hence the segment stabilizers for the action of  $\Gamma$  on  $X_\infty$  must be virtually cyclic. This completes the proof of Paulin's theorem.  $\square$

### SECTION 3: CONVEX HULLS

A subset  $\Sigma$  of a geodesic metric space  $X$  is said to be *geodesically convex* if for all  $p, q \in \Sigma$  every geodesic segment from  $p$  to  $q$  is completely contained in  $\Sigma$ . Given a bounded set  $Y \subset X$ , perhaps the most natural way to define its convex hull is as the intersection of all geodesically convex sets containing  $Y$ .

If  $X$  is simply connected and non-positively curved then round balls are geodesically convex and hence the convex hull of a bounded set is bounded. However, for more general geodesic metric spaces, even  $\delta$ -hyperbolic spaces, it may happen that the convex hull of a finite set is the whole of the ambient space  $X$ . The following example illustrates how general this problem is.

**3.1 PROPOSITION.** *Given any finitely generated group  $\Gamma$  there exists a finite generating set  $S$  and a finite subset  $Y \subset \Gamma$  such that the convex hull of  $Y$  in the Cayley graph  $X(\Gamma, S)$  is the whole of  $X(\Gamma, S)$ .*

*Proof.* Let  $A$  be any finite generating set for  $\Gamma$ , and take  $S$  to be the set of those elements of  $\Gamma$  which are a distance 1 or 4 from the identity in the Cayley graph of  $\Gamma$  with respect to  $S$ . Let  $Y$  be the set of elements of  $\Gamma$  which are a distance at most 3 away from the identity in the Cayley graph associated to  $A$ .