Introduction

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ON HAUSDORFF-GROMOV CONVERGENCE AND A THEOREM OF PAULIN

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ABSTRACT. We give an elementary account of ideas related to Hausdorff-Gromov convergence and explain how, among other things, these ideas can be used to prove a theorem of F. Paulin: If a group Γ is word hyperbolic and its outer automorphism group is infinite, then Γ acts by isometries on an **R**-tree with virtually cyclic segment stabilizers and no global fixed points.

INTRODUCTION

The purpose of this article is to give an essentially self-contained proof of the following theorem of F. Paulin. (The technical terms appearing in this theorem are explained below.)

THEOREM (Paulin). *If Γ is a word hyperbolic group and $Out(\Gamma)$ is infinite, then Γ acts by isometries on an* **R***-tree with virtually cyclic segment stabilizers and no global fixed points.*

We feel that this theorem and (more especially) the techniques involved in its proof are central to the study of word hyperbolic groups and related topics. This is illustrated, for example, by the variety of ways in which these ideas have entered the work of Rips and Sela. The techniques in question centre on Gromov's generalisation of Hausdorff convergence, as developed in Paulin's thesis and Bestvina's work on degeneration of hyperbolic structures. In light of the continuing importance of these techniques, it seemed to us desirable that an elementary and self-contained account of them should be made available.

Let us recall the definitions of the terms appearing in the statement of Paulin's theorem. Let X be a metric space. A topological arc in X is called

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a geodesic segment if, with the induced metric, it is isometric to a compact interval of the real line. X is said to be a geodesic space if every pair of points in X can be joined by a geodesic segment. A geodesic triangle in X consists of three points (vertices) together with a choice of geodesic segment (side) joining each pair of them. A geodesic triangle is said to be δ-slim if each of its sides is contained in the δ-neighbourhood of the other two. A geodesic metric space X is said to be δ-hyperbolic if every geodesic triangle in X is δ-slim. An **R**-tree is a 0-hyperbolic space, in other words, a geodesic metric space in which every geodesic triangle is degenerate, i.e., is a tripod. The most primitive example of an **R**-tree is an ordinary simplicial tree in which each of the edges is metrized so as to have length 1. For many purposes, particularly in this article, it is useful to think of a δ-hyperbolic space as a somewhat thickened version of an **R**-tree and to keep in mind the idea that if one looks at the space from so far away that distances of the order of δ appear negligible, then a δ-hyperbolic space takes on the appearance of an **R**-tree.

Recall that the Cayley graph $X(\Gamma, S)$ of a group Γ with respect to a choice of finite generating set $S \subset \Gamma - \{e\}$ is the metric graph whose vertex set is Γ and which has one edge of unit length joining $\gamma \in \Gamma$ to γs whenever $s \in S$. A group is said to be *word hyperbolic* if its Cayley graph $X(\Gamma, S)$ is δ-hyperbolic for some δ. (The hyperbolicity of $X(\Gamma, S)$, but not the specific value of δ, is independent of the choice of S — see [GH] or [Sho].) The class of word hyperbolic groups was introduced by Gromov in [G2], and has proved to be a fruitful context in which to extend many elegant results of hyperbolic geometry, particularly results about geometrically finite groups of isometries of real hyperbolic space that do not contain any parabolic elements.

This article is organised as follows. In Section 1 we describe some basic facts about various generalisations of Hausdorff convergence. In the proof of such elementary facts one discerns a general pattern of argument that can be applied more generally, and it is this pattern, rather than specific compactness criteria, that seems to be most useful in a wider context. In Section 2 we illustrate this point in proving Paulin's theorem.

Paulin has developed an equivariant version of Hausdorff-Gromov convergence, which he calls Gromov convergence, and has used this to give elegant formulations of compactness theorems of Thurston and Culler-Morgan. The compactness criterion which he originally developed in this context relies upon the existence of convex hulls in the spaces under consideration; in spaces such as the Cayley graph of a word hyperbolic group one cannot in general form a precise convex hull for finite sets. This difficulty is the subject of Section 3. Section 4 contains some concluding remarks and a brief discussion of recent work which draws on ideas similar to those discussed in this article.

SECTION 1: HAUSDORFF-GROMOV CONVERGENCE

Until further notice, we fix a compact metric space X and denote by $\mathscr{C}(X)$ the set of closed subsets of X. We shall always denote the open ε-neighbourhood in X of $A \subset X$ by $V_\varepsilon(A)$.

The starting point for our discussion is the following classical construction.

1.1 DEFINITION. *The Hausdorff metric on* $\mathscr{C}(X)$ *is defined by:*

$$D(A, B) = \inf\{\varepsilon \mid A \subseteq V_\varepsilon(B) \quad \textit{and} \quad B \subseteq V_\varepsilon(A)\} .$$

1.2 PROPOSITION. *D is indeed a metric and $\mathscr{C}(X)$ equipped with this metric is compact.*

Proof. The only nontrivial point to check is that $\mathscr{C}(X)$ is compact.

Consider a sequence C_i in $\mathscr{C}(X)$. We must exhibit a convergent subsequence. First notice that given any $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that, in its induced metric from X, every $A \in \mathscr{C}(X)$ can be covered by $N(\varepsilon)$ open balls of radius ε. Indeed, because X is compact one can cover it with $N(\varepsilon)$ balls of radius $\varepsilon/2$, then for each such ball which intersects A one chooses a point in the intersection and takes the ball of radius ε about that point. Thus for every positive integer n and every C_i, by taking duplicates if necessary, we may assume that C_i is covered by precisely $N(1/n)$ balls of radius $1/n$, with centres $x_n(i,j)$ for $j = 1, \ldots, N(1/n)$. Furthermore, it is clear from our description of how to choose the $x_n(i,j)$ that this can be done so as to ensure that $x_{n+1}(i,j) = x_n(i,j)$ if $j \leqslant N(1/n)$, thus we may drop the subscript n.

At this stage we have constructed sequences of points $\{x(i,j)\}_j \subset C_i$, each of which has the property that for all $n \in \mathbf{N}$ the balls of radius $1/n$ about the first $N(1/n)$ terms in the sequence cover C_i.

$$\begin{array}{l} C_1 \ni x(1, 1), x(1, 2), \ldots, x(1,j), \ldots \\ C_2 \ni x(2, 1), x(2, 2), \ldots, x(2,j), \ldots \\ \quad\vdots \\ C_i \ni x(i, 1), x(i, 2), \ldots, x(i, j), \ldots \\ \quad\vdots \end{array}$$