

## 2.2. Problems

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cients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

## 2.2. PROBLEMS

i) Since for fixed  $k$  the dimension of  $J_{k,m}$  grows linearly in  $m$ , the map  $\rho_m$  defined by (3) for  $m \gg_k 0$  cannot be surjective. Is there any simple or nice description of the image of  $\rho_m$  or  $(im \rho_m | S_k(\Gamma_2))^\perp$ ? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on  $\Gamma_2$  which generate  $S_k(\Gamma_2)$ , as certain infinite linear combinations of Poincaré series on  $\Gamma_1^J$  [22]. Taking scalar products one obtains a characterization of  $(im \rho_m | S_k(\Gamma_2))^\perp$  as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that  $\rho_1$  is surjective).

ii) A skew-holomorphic Jacobi form of weight  $k \in \mathbf{Z}$  and index  $m \in \mathbf{N}_0$  on  $\Gamma_1^J$  as introduced by Skoruppa is a complex-valued  $C^\infty$ -function  $\phi(\tau, z)$  ( $\tau \in \mathcal{H}$ ,  $z \in \mathbf{C}$ ) satisfying the following properties: 1)  $\phi$  is holomorphic in  $z$  and is annihilated by the heat operator  $8\pi im \partial / \partial \tau - \partial^2 / \partial z^2$ ; 2)  $\phi$  satisfies the same transformation formula under  $\Gamma_1^J$  as a holomorphic Jacobi form of weight  $k$  and index  $m$  (cf. § 1.2) except that the factor  $(c\tau + d)^k$  has to be replaced by  $(c\bar{\tau} + d)^{k-1} |c\tau + d|$ ; 3)  $\phi$  has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \geq 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \quad (v = \text{Im}(\tau)).$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of  $F(Z)$  w.r.t.  $e^{2\pi i \tau'}$  (where as usual  $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ ) not only depend on  $\tau$  and  $z$  but also on  $\text{Im}(\tau')$ , and it is a priori not obvious how to get rid of the latter variable and to produce “true” Jacobi forms.

Let  $k$  be an odd integer and denote by  $M_{1/2, k-1/2}(\Gamma_2)$  the space of Siegel-Maass wave forms “of type  $(1/2, k-1/2)$ ” as defined in [26], i.e. the space of real-analytic functions  $F: \mathcal{H}_2 \rightarrow \mathbf{C}$  which satisfy

$$F(M \langle Z \rangle) = \det(C\bar{Z} + D)^{k-1} | \det(CZ + D) | F(Z)$$

for all  $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$  and which are annihilated by the matrix differential operator

$$\Omega_{1/2, k-1/2} := (Z - \bar{Z}) \left( (Z - \bar{Z}) \frac{\partial}{\partial Z} \right)' \frac{\partial}{\partial \bar{Z}} + \frac{1}{2} (Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}} - \left( k - \frac{1}{2} \right) (Z - \bar{Z}) \frac{\partial}{\partial Z}$$

$$\text{where } \frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial \tau} & \frac{1}{2} \frac{\partial}{\partial z} \\ \frac{1}{2} \frac{\partial}{\partial \bar{z}} & \frac{\partial}{\partial \tau'} \end{pmatrix}$$

and  $\frac{\partial}{\partial \bar{Z}}$  is defined analogously (the notation “of type  $(1/2, k - 1/2)$ ” comes from the fact that the factor of automorphy of  $F$  can be written as  $\det(C\bar{Z} + D)^{k-1/2} \det(CZ + D)^{1/2}$  with appropriate choice of the square root).

Using certain invariance properties of  $\Omega_{1/2, k-1/2}$  under the action of  $\text{Sp}_2(\mathbf{R})$  one can define Hecke operators  $T_n$  ( $n \in \mathbf{N}$ ) on  $M_{1/2, k-1/2}(\Gamma_2)$  in the usual way. Let

$$E_{1/2, k-1/2}^{(2)}(Z) := \sum_{(C, D)} \det(CZ + D)^{-k+1} |\det((CZ + D))|^{-1} \quad (k > 3)$$

be the Maass-Siegel-Eisenstein series in  $M_{1/2, k-1/2}(\Gamma_2)$  ([26; 27, §18]; summation over all pairs  $(C, D)$  of relatively prime symmetric  $(2, 2)$ -matrices inequivalent under left-multiplication by  $GL_2(\mathbf{Z})$ ). Then the following can be shown:

- 1) The function  $E_{1/2, k-1/2}^{(2)}$  is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to  $\zeta(s - k + 1) \zeta(s - k + 2) L_{E_{2k-2}}(s)$  where  $E_{2k-2}$  is the normalized Eisenstein series of weight  $2k - 2$  on  $\Gamma_1$  (this implies that  $E_{1/2, k-1/2}^{(2)}$  for all primes  $p$  is annihilated by the Hecke operator  $\mathcal{E}_p$  defined analogously as in (5));
- 2) if  $e_{1/2, k-1/2; m}(\tau, z, \text{Im}(\tau'))$  is the  $m$ -th Fourier-Jacobi coefficient of  $E_{1/2, k-1/2}^{(2)}$  and if for  $m > 0$  one carries out a similar limit process as in [19, §2, Remark ii) after the proof of Thm. 1], i.e. essentially replaces  $\text{Im}(\tau')$  by  $(\text{Im}(z))^2 / \text{Im}(\tau) + \delta$  and lets  $\delta \rightarrow \infty$ , then one obtains a skew-holomorphic Eisenstein series of weight  $k$  and index  $m$  (in fact, finite linear combinations of such Eisenstein series if  $m$  is not squarefree).

The following questions therefore are suggestive:

- 1) if one starts with an arbitrary  $F \in M_{1/2, k-1/2}(\Gamma_2)$ , does the above limit process produce skew-holomorphic Jacobi forms of weight  $k$ ?
- 2) define  $M_{1/2, k-1/2}^*(\Gamma_2)$  as the subspace of  $M_{1/2, k-1/2}(\Gamma_2)$  consisting of the intersection of the kernels of the operators  $\mathcal{E}_p$  for all primes  $p$ . Does there exist a natural map  $V$  from skew-holomorphic Jacobi forms of weight  $k$  and index 1 to  $M_{1/2, k-1/2}^*(\Gamma_2)$  similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on  $\mathrm{Sp}_2$ . It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus  $n > 2$  has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for  $n \geq 33$  the map which sends a Siegel modular form of weight 16 on  $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$  to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for  $n = 3$  due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform  $F$  of even integral weight  $k$  on  $\Gamma_3$  could be constructed from a pair  $(f, g)$  of elliptic Hecke eigenforms of weights  $(k_1, k_2)$  equal to  $(k, 2k - 4)$  or  $(k - 2, 2k - 2)$  such that the (formal) spinor zeta function of  $F$  should be equal to  $L_f(s - k_2/2)L_f(s - k_2/2 + 1)L_{f \otimes g}(s)$  where  $L_{f \otimes g}(s)$  essentially is the Rankin convolution of  $f$  and  $g$  ([*loc. cit.*, §4]; note that for  $n > 2$  the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on  $\Gamma_n$  is not known).

### §3. SPINOR ZETA FUNCTIONS

#### 3.1. RESULTS

Although the Maass space  $S_k^*(\Gamma_2)$  as discussed in the previous section is an important subspace of  $S_k(\Gamma_2)$  in its own right, one quickly realizes that the "true" Siegel cusp forms on  $\Gamma_2$  should lie in the orthogonal complement of  $S_k^*(\Gamma_2)$  (cf. Theorem 2 in §2 and its discussion). It is therefore even more