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MAXIMALLY COMPLETE FIELDS

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ABSTRACT. Kaplansky proved in 1942 that among all fields with a valuation having a given divisible value group G, a given algebraically closed residue field R, and a given restriction to the minimal subfield (either the trivial valuation on $\mathbf{Q}$ or $\mathbf{F}_p$, or the p-adic valuation on $\mathbf{Q}$), there is one that is maximal in the strong sense that every other can be embedded in it. In this paper, we construct this field explicitly and use the explicit form to give a new proof of Kaplansky's result. The field turns out to be a Mal'cev-Neumann ring or a p-adic version of a Mal'cev-Neumann ring in which the elements are formal series of the form $\sum_{g \in S} \alpha_g p^g$ where S is a well-ordered subset of G and the α_g's are residue class representatives. We conclude with some remarks on the p-adic Mal'cev-Neumann field containing $\bar{\mathbf{Q}}_p$.

I. INTRODUCTION

It is well known that if k is an algebraically closed field of characteristic zero, then the algebraic closure of the field of Laurent series $k((t))$ is obtained by adjoining $t^{1/n}$ for each integer $n \geqslant 1$, and that the expansion of a solution to a polynomial equation over $k((t))$ can be obtained by the method of successive approximation. (For example, to find a square root of $1 + t$, one solves for the coefficients of $1, t, t^2, \ldots$ in turn.) But if k is algebraically closed of characteristic p, $\cup_{n=1}^{\infty} k((t^{1/n}))$ is no longer an algebraic closure of $k((t))$. In particular, the Artin-Schreier equation $x^p - x = t^{-1}$ has no solution in $\cup_{n=1}^{\infty} k((t^{1/n}))$. (See p. 64 of Chevalley [3].) If one attempts nevertheless to successively approximate a solution, one obtains the expansion (due to Abhyankar [1])

$$x = t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \cdots ,$$

in which the exponents do not tend to ∞, as they should if the series were to converge with respect to a valuation in the usual sense. However, one checks