5. $\bar{W}$ IS PATH CONNECTED

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5. $\overline{W}$ IS PATH CONNECTED

Here we refine the argument of the previous section to prove $\overline{W}$ is path connected. There are two main difficulties that arise. One is that the path connected analogue of Lemma 4.2, although still true (at least when M is Hausdorff), is much harder to prove. The second is that a decreasing intersection of compact path connected sets need not be path connected, so we can no longer restrict our attention to the zeros within $\{z : |z| < 1 - \delta\}$.

The lifting lemma below will be used as a substitute for Lemma 4.2. Its proof is based on proofs obtained independently by David desJardins and Emanuel Knill.

LEMMA 5.1. (Lifting lemma): *Let M be a Hausdorff space and let $\pi : M^n \to M^n/S_n$ be the projection map. Let $f : [0, 1] \to M^n/S_n$ be a continuous map. Then there is a continuous map $g : [0, 1] \to M^n$ such that $f = \pi \circ g$.*

SUBLEMMA 5.1. *Let $\Delta = \{t \in [0, 1] : f(t)$ consists of n copies of a single point$\}$. Let $g : [0, 1] \to M^n$ be an arbitrary function that is a lift of f. Then g is automatically continuous at all $t_0 \in \Delta$.*

Proof. Suppose $t_0 \in \Delta$ and $f(t_0) = \{x, x, \ldots, x\}$. If U is an open neighborhood of x,

$$g^{-1}(U^n) = f^{-1}(\pi(U^n))$$

which is open. Since such subsets U^n form a neighborhood base at $(x, x, \ldots, x) \in M^n$, this proves that g is continuous at t_0.

SUBLEMMA 5.2. *Let I_1, I_2 be closed subintervals of $[0, 1]$ such that $I_1 \cap I_2$ is a single point $\{t\}$. If g_j is a continuous lift of f on $I_j (j = 1, 2)$ then there is a continuous lift g of f on $I_1 \cup I_2$ such that $g|_{I_1} = g_1$.*

Proof. Since $g_1(t)$ and $g_2(t)$ differ only by a permutation, we can compose g_2 with a permutation $\sigma : M^n \to M^n$ and then paste the result to g_1.

SUBLEMMA 5.3. *The conclusions of Sublemma 5.2 hold even if I_1 and I_2 intersect in more than a point.*

Proof. This follows form Sublemma 5.2 since $I_1 \cup I_2$ can be expressed as the union of I_1 with at most two closed subintervals of I_2 each meeting I_1 in a point.

SUBLEMMA 5.4. *If I is a closed subinterval of $[0, 1]$ and every $t \in I$ has a neighborhood on which f has a lift, then f has a lift on I.*

Proof. By compactness, we can cover I by closed intervals $I_1, I_2, \ldots, I_k$ on which f has a lift, and we may assume $I_j \cap I_{j+1} \neq \varnothing$ for $1 \leqslant j \leqslant k$. By induction on j, Sublemma 5.3 lets us extend the lift on I_1 to a lift on $I_1 \cup I_2 \cup \cdots \cup I_j$.

SUBLEMMA 5.5. *The same holds if I is any subinterval of $[0, 1]$.*

Proof. Let $C_1 \subseteq C_2 \subseteq \cdots$ be closed intervals such that $\bigcup_{i=1}^{\infty} C_i = I$. By Sublemma 5.4, there is a lift on each C_i. By repeated use of Sublemma 5.3, extend the lift on C_1 to a lift on C_2, extend this to C_3, etc. This process gives a lift on I.

Proof of Lemma 5.1. We use induction on n. The case $n = 1$ is trivial, so assume $n > 1$. By Sublemma 5.1, it suffices to find a lift on each connected component I of $[0, 1]\backslash\Delta$. By Sublemma 5.5 it suffices to show that any $t_0 \in I$ has a neighborhood on which there is a lift.

Suppose $z_1, z_2, \ldots, z_k (k \geqslant 2)$ are the distinct elements of the multiset $f(t_0)$, occurring with multiplicities $n_1, n_2, \ldots, n_k$ respectively. Since M is Hausdorff, there exist pairwise disjoint neighborhoods U_i of z_i. Let N be a closed interval neighborhood of t_0 such that $t \in N$ implies $f(t) \in \pi(U_1^{n_1} \times \cdots \times U_k^{n_k})$. Then on N, we can lift f to a path $\tilde{f}$ in $M^{n_1}/S_{n_1} \times \cdots \times M^{n_k}/S_{n_k}$ since the projection

$$M^{n_1}/S_{n_1} \times \cdots \times M^{n_k}/S_{n_k} \to M^n/S_n$$

restricts to a homeomorphism on the projections of $U_1^{n_1} \times \cdots \times U_k^{n_k}$. By the inductive hypothesis applied to each of the k coordinates of $\tilde{f}$, we can lift $\tilde{f}$ to a path in $M^{n_1} \times \cdots \times M^{n_k} = M^n$ as desired. □

THEOREM 5.1. *$\bar{W}$ is path connected.*

Proof. Let M be $\{z : |z| \leqslant 1\}$ with the unit circle shrunk to a point P. Again M is topologically a sphere, so we may give it a bounded metric d.

Let M^∞ be the set of sequences $x = \{x_i\}_{i=1}^\infty$ which converge to P and define a metric d_∞ on M_∞ by

$$d_\infty(x, y) = \sup_i d(x_i, y_i) .$$

Let the group S_∞ of permutations of $\{1, 2, \ldots,\}$ act on M^∞ by permuting the coordinates. Define a metric D on the quotient space M^∞/S_∞ by letting

$$D(\bar{x}, \bar{y}) = \inf_{\sigma \in S_\infty} d_\infty(x, \sigma y) .$$

Here $(\bar{x}, \bar{y})$ denote the projections of $x, y \in M^\infty$ to M^∞/S_∞. (That $D(\bar{x}, \bar{y}) = 0$ if and only if $\bar{x} = \bar{y}$ requires the convergence of x, y.) The set of zeros of a power series

$$1 + \varepsilon_1 z + \varepsilon_2 z^2 + \cdots$$

inside $\{z : |z| < 1\}$ forms a sequence in M converging to P (by Proposition 2.1) or else is finite, in which case we append an infinite sequence of P's. This defines a map

$$f : \{0, 1\}^\omega \to M^\infty/S_\infty .$$

By the same Rouché's theorem argument used in the proof of Theorem 4.1, this map is continuous. The conditions of Lemma 4.1 hold for the same reason as before, so the image of f is path connected.

Suppose $z_0 \in \bar{W} \cap \{z : |z| < 1\}$. Let $\omega : [0, 1] \to M^\infty/S_\infty$ be a path from the image under f of a 0, 1 power series vanishing at z_0 to $f((0, 0, \ldots)) = \{P, P, P, \ldots\}$.

Fix $m \geqslant 1$, and let M_m be $\{z : |z| \leqslant 1\}$ with the annulus

$$\{z : 1 - 1/m \leqslant |z| \leqslant 1\}$$

shrunk to a point Q. Define $|\ |$ on M_m by letting $|Q| = 1 - 1/m$. By Proposition 2.1 there is an upper bound n on the number of zeros of a 0, 1 power series inside $\{z : |z| < 1 - 1/m\}$. The path ω induces a path

$$\omega_m : [0, 1] \to (M_m)^n/S_n .$$

(Apply the projection $M \to M_m$ to each element of $\omega(t)$, and throw away infinitely many Q's to get $\omega_m(t)$.)

Pick $m_0 \geqslant 1$ such that $|z_0| < 1 - 2/m_0$. We define inductively a sequence of paths

$$\tilde{\omega}_m : [0, t_m] \to \bar{W} , \qquad m = m_0, m_0 + 1, \ldots ,$$

each extending the one before. First apply Lemma 5.1 to lift ω_{m_0} to a path $[0, 1] \to M^n_{m_0}$. Since some coordinate of $\omega_{m_0}(0)$ is z_0 and since all coordinates of $\omega_{m_0}(1)$ are Q, we get a path $\tilde{\omega}_m$ from z_0 to Q in M_{m_0}. Let t_{m_0} be the smallest $t \in [0, 1]$ such that $|\tilde{\omega}_{m_0}(t)| \geqslant 1 - 2/m_0$. Then by restriction to $[0, t_{m_0}]$ we get a path $\tilde{\omega}_{m_0}$ in $\mathbf{C}$ since $\{z \in M_{m_0} : |z| \leqslant 1 - 2/m_0\}$ can be identified with $\{z \in \mathbf{C} : |z| \leqslant 1 - 2/m_0\}$. Finally, since $\tilde{\omega}_{m_0}(t)$ is always a coordinate of $\omega(t)$, $\tilde{\omega}_{m_0}(t) \in \bar{W}$ for all $t \in [0, t_{m_0}]$.

By the same process, we inductively find for each $m > m_0$ a path $\tilde{\omega}_m : [t_{m-1}, 1] \to M_m$ such that $\tilde{\omega}_m(t_{m-1}) = \tilde{\omega}_{m-1}(t_{m-1}$. Let t_m be the smallest $t \geqslant t_{m-1}$ such that

$$|\tilde{\omega}_m(t)| \geqslant 1 - \frac{2}{m},$$

and obtain a path

$$\tilde{\omega}_m : [t_{m-1}, t_m] \to \bar{W}$$

which we append to $\tilde{\omega}_{m-1}$ to obtain

$$\tilde{\omega}_m : [0, t_m] \to \bar{W}$$

such that $\tilde{\omega}_m(t)$ is always a coordinate of $\omega(t)$.

Let $t_\infty = \sup_m t_m$. Piecing together the $\tilde{\omega}_m$'s gives a continuous map

$$\tilde{\omega} : [0, t_\infty) \to \bar{W}$$

such that $\tilde{\omega}(t)$ is a coordinate of $\omega(t)$ for all $t \in [0, t_\infty)$. The set of limit points of $|\tilde{\omega}(t)|$ as $t \to t_\infty$ is a closed interval I. Let $\omega(t_\infty) = \{z_1, z_2, z_3, \ldots\}$. If $r \in [0, 1)$ is distinct from $|z_1|, |z_2|, |z_3|, \ldots$ then $\sup_i |r - |z_i|| > 0$ (since $\{z_i\} \to P$) and by continuity of ω, r also differs by some ε from all coordinates of $\omega(t)$ for t in a neighborhood of t_∞, so r cannot be a limit point of $|\tilde{\omega}(t)|$ as $t \to t_\infty$. Thus $I \subseteq \{1, |z_1|, |z_2|, \ldots\}$ but $|z_i| \to 1$ so I must be a single point. Since $|\tilde{\omega}(t_m)| = 1 - 2/m$ for $m \geqslant m_0$, $\lim_{t \to t_\infty} |\tilde{\omega}(t)| = 1$.

Case 1: 1 is the only limit point of $\tilde{\omega}(t)$ as $t \to t_\infty$. Then $\tilde{\omega}$ extends to a path $[0, t_\infty] \to \bar{W}$ from z_0 to 1.

Case 2: There is a limit point $\theta \neq 1, |\theta| = 1$, of $\tilde{\omega}(t)$ as $t \to t_\infty$. By Theorem 3.1, there is an open disc centered at θ contained in $\bar{W}$. For some $t < t_\infty$, $\tilde{\omega}(t)$ is in this disc, so we can replace the tail end of $\tilde{\omega}$ on $[t, t_\infty)$ by a straight line from $\tilde{\omega}(t)$ to θ in $\bar{W}$.

In either case we can connect z_0 to a point on the unit circle via a path in $\bar{W}$. The same is true if $z_0 \in \bar{W}$, $|z_0| > 1$, since $\bar{W}$ is closed under $z \mapsto 1/z$. Since $\bar{W}$ contains the unit circle, this proves that $\bar{W}$ is path connected. □