III. Lifting to C^3

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of C belongs to the interior of the polynomial hull of $\bar{W}$, *then C is entirely included in the interior of the polynomial hull of* $\bar{W}$.

By holomorphic curve we will mean a connected 1-dimensional holomorphic manifold.

Proof. Let O be the interior of the polynomial hull of $\bar{W}$. It has to be shown that the set of points $p \in C$ which belong to O is closed in C. It is obviously open. Things being so localized one has to face the following situation: a "small" analytic disk given by a holomorphic parametrization $\varphi : \bar{\Delta} \to C$ (Δ the unit disk in $\mathbf{C}$) so that $\varphi(1) \in O$, U^+ a side of M included in W (at least one of the two sides is such) hence in O, in some neighborhood of $\varphi(\bar{\Delta})$; and one has to show that $\varphi(0) \in O$. Fix ψ a holomorphic map from $\mathbf{C}$ into $\mathbf{C}^n$ so that: $\psi(e^{i\theta}) \simeq -\vec{N}$ for θ outside some small neighborhood of 0 (mod 2π), where $\vec{N}$ is the unit outer normal to M (with respect to U^+), at say the point $\varphi(0)$, and $\psi(0)$ is arbitrarily chosen.

For $\eta > 0$, η small enough $\varphi(e^{i\theta}) + \eta\psi(e^{i\theta}) \in O$ for all θ, hence $\varphi(0) + \eta\psi(0) \in O$. Taking into account some uniformity with respect to $\psi(0)$, this gives Lemma 2.

III. LIFTING TO $\mathbf{C}^3$

We are simply going to consider sets K in $\mathbf{C}^3$ rotationally invariant in the first variable, that we describe as follows. For each $t \in [0, t_0]$ we are given a compact set $K_t \subset \mathbf{C}^2$. We consider the set $K \subset \mathbf{C}^3$ which is the closure of the set $\{(w, z_1, z_2) \in \mathbf{C}^3; (z_1, z_2) \in K_{|w|}, |w| \leqslant t_0\}$. i.e.

$$K = \overline{\bigcup_{|w| \leqslant t_0} \{w\} \times K_{|w|}}.$$

$\hat{K}$ denotes the polynomial hull of K in $\mathbf{C}^3$, while $\widehat{\cup K_t}$ denotes the polynomial hull in $\mathbf{C}^2$ of the closure of the set $\bigcup_{t \leqslant t_0} K_t$.

LEMMA 3. *Let* $(0, \zeta_1, \zeta_2) \in \mathbf{C}^3$, *the following are equivalent:*

$$\begin{cases} (i) & (0, \zeta_1, \zeta_2) \in \hat{K} \\ (ii) & (\zeta_1, \zeta_2) \in \widehat{\cup K_t}. \end{cases}$$

Proof. (i) $\Rightarrow$ (ii) is trivial. We are interested in (ii) $\Rightarrow$ (i). Let $P(w, z_1, z_2)$ be a polynomial in 3 variables. To P we associate the polynomial $\tilde{P}$ defined by

$$\tilde{P}(w, z_1, z_2) = P(0, z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta} w, z_1, z_2) d\theta .$$

Since K is invariant under rotation in the w variable:

$$\sup_K |\tilde{P}| \leqslant \sup_K |P| .$$

Set $Q(z_1, z_2) = P(0, z_1, z_2)$. Using (ii) one gets

$$|P(0, \zeta_1, \zeta_2)| = |Q(\zeta_1, \zeta_2)| \leqslant \sup_{\cup K_t} |Q| = \sup_K |\tilde{P}| \leqslant \sup_K |P| .$$

So (i) is established.

Remark. There is another approach to Lemma 3, which may better "explain" the situation, and that we just sketch. If $\varphi: \Delta \to \mathbf{C}^2$ is a holomorphic disk (φ continuous on $\bar{\Delta}$, holomorphic on Δ) and T is a continuous map from $\mathbf{R}/2\pi\mathbf{Z}$ into $[0, t_0]$ so that $\varphi(e^{i\theta}) \in K_{T(\theta)}$ $(\theta \in [0, 2\pi))$, then $\varphi(0) \in \widehat{\cup K_t}$. One sees that $(0, \varphi(0)) \in \hat{K}$ by considering holomorphic disks $(Q, \varphi): \Delta \to \mathbf{C} \times \mathbf{C}^2$, with $Q(0) = 0$ and $|Q(e^{i\theta})| \simeq T(\theta)$. Carrying this out in general may require the use of the fundamental theorem by Poletsky [6], which says that, in an appropriate sense, polynomial hulls are always explained by holomorphic disks.

IV. Trepeau's example

Here we describe a class of examples. Let χ be a smooth real valued function defined on [0, 1], constant in no neighborhood of 0, and so that $\chi(0) = 0$, $|\chi| < 1$. In one of the versions of Trepreau's original example $\chi(t) = t$. Let $\mathscr{M}$ be the generic 4-dimensional manifold in $\mathbf{C}^3$, given by:

$$\mathscr{M} = \{(w, z_1, z_2) \in \mathbf{C}^3, |w| < 1, z_1 = s_1 + i\chi(|w|^2)s_2 ,$$
$$z_2 = s_2 - i\chi(|w|^2)s_1;\ (s_1, s_2) \in \mathbf{R}^2\} .$$

Notice that on $\mathscr{M}$, $z_1^2 + z_2^2$ is a real valued function, (on $\mathscr{M}$, $z_1^2 + z_2^2 \geqslant 0$), hence:

(*) *Any function which depends only on* $(z_1^2 + z_2^2)$ *is a CR function on* $\mathscr{M}$.

This already gives example of *CR* functions which cannot be holomorphically extended to any wedge. The existence of such functions is related to the fact that $\mathscr{M}$ is not "minimal" (in the sense of Tumanov), it contains $\mathbf{C} \times \{0\} \times \{0\}$ as a (nongeneric) *CR* manifold of same *CR* dimension (see [9], [2]).