5. More questions

Finally

$$\liminf_{\delta(x)\to\infty} \frac{\delta(Tx)}{\delta(x)} = \frac{1}{\Theta(\Delta)}\,.$$

The following table gives the first values of Θ

n	1	2	3	4	5	6	7	8	9	10	...
$\Theta(n)$	1	3	3	3	5	3	5	5	5	5	...

Actually Theorem 1 can be improved. There exist two constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that for all rational x

$$\frac{1}{\Theta(\Delta)}\,\delta(x) - C_1 \leqslant \delta(Tx) \leqslant \Theta(\Delta)\,\delta(x) + C_2\,.$$

Both inequalities are sharp apart from the exact values of C_1 and C_2.

5. More questions

To every Möbius map T we associate the interval $I(T) = [\Theta^{-1}(\Delta), \Theta(\Delta)]$.

Problem 3. *Is it true that for all* $\zeta \in I(T)$ *there exists a sequence of rational numbers* x_n *such that* $\lim_{n\to\infty} \delta(x_n) = \infty$ *and*

$$\lim_{n\to\infty} \frac{\delta(Tx_n)}{\delta(x_n)} = \zeta\,?$$

Problem 4. *Let* $T_1, T_2, \cdots, T_k$ *be Möbius maps with pairwise coprime determinants* $\Delta_1, \Delta_2, \cdots, \Delta_k$.

Is it true that for all

$$(\zeta_1, \zeta_2, \cdots, \zeta_k) \in \prod_{i=1}^{k} I(T_i)$$

there exists a sequence of rational x_n *with strictly increasing depths such that for all* $i = 1, 2, \cdots, k$

$$\lim_{n\to\infty} \frac{\delta(T_i x_n)}{\delta(x_n)} = \zeta_i\,?$$

Can k *be infinite?*

The following result should be mentioned at this point.

THEOREM 3. *There exists a sequence of rational numbers* x_n *with strictly increasing depths such that for all Möbius maps* T

$$\lim_{n \to \infty} \frac{\delta(Tx_n)}{\delta(x_n)} = 1 .$$

The proof is quite simple. To each irrational

$$x = [c_0, c_1, c_2, \cdots]$$

we associate the sequence of best approximations

$$x_n = \frac{p_n}{q_n} = [c_0, c_1, c_2, \cdots, c_n] .$$

Paul Lévy [9] showed that for almost all x

$$\ln q_n \sim \frac{\pi^2}{12 \ln 2} n$$

as n goes to infinity (see for example [1] p. 45). In other words, for almost all x

$$\delta(x_n) \sim \frac{12 \ln 2}{\pi^2} \ln q_n .$$

Therefore, for almost all x

$$\delta\left(\frac{ax_n + b}{cx_n + d}\right) \sim \frac{12 \ln 2}{\pi^2} \ln(cp_n + dq_n) .$$

Now $p_n \sim xq_n$ so that

$$cp_n + dq_n \sim (cx + d) q_n$$
$$\ln(cp_n + dq_n) \sim \ln q_n .$$

Hence for almost all x

$$\delta\left(\frac{ax_n + b}{cx_n + d}\right) \sim \delta(x_n) .$$

By countable intersection, we conclude that for almost all x and for all Möbius map T

$$\delta(Tx_n) \sim \delta(x_n) . \qquad \text{QED}$$

PROBLEM 5. *Let* T *be a given Möbius map and let* $I(T)$ *be the associated interval. Let* $\zeta \in I(T)$. *To compute the Hausdorff dimension of those* x *for which*

$$\lim_{n \to \infty} \frac{\delta(Tx_n)}{n} = \zeta .$$

Extend this problem to higher dimensions in the spirit of problem 4.

§6. QUADRATIC SURDS

Let x be a real quadratic number. Its continued fraction expansion is ultimately periodic. Let $\pi(x)$ be its period. H. Cohen [3], followed by J. Cusick [4] and Paysant-Leroux [11] studied the action of a Möbius map on the period. They established that

$$\lim_{\pi(x) \to \infty} \sup \frac{\pi(Tx)}{\pi(x)} = R(\Delta)$$

where $R(\Delta)$ is an integer. Furthermore

$$A\, n \ln n \leqslant R(n) \leqslant B\, n \ln n + 1$$

for some constants $A > 0$, $B > 0$. A simple argument then shows that

$$\lim_{\pi(x) \to \infty} \inf \frac{\pi(Tx)}{\pi(x)} = \frac{1}{R(\Delta)} .$$

PROBLEM 6. *Is it true that for all real quadratic irrational* x

$$\sup_n \pi(x^n) = \infty \, ?$$

Define the interval

$$J(\Delta) = \left[\frac{1}{R(\Delta)}, R(\Delta)\right] .$$

PROBLEM 7. *Let* $\zeta \in J(\Delta)$. *Prove the existence of a sequence of real quadratic numbers* x_n *with strictly increasing period such that*

$$\lim_{n \to \infty} \frac{\pi(Tx_n)}{\pi(x_n)} = \zeta .$$

Extend this result to higher dimensions as in Problem 4.