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REMARKS AND PROBLEMS ON FINITE AND PERIODIC CONTINUED FRACTIONS

by Michel MENDÈS FRANCE

SUMMARY. We present eight problems related to the length of continued fractions of rational numbers and to the length of the period of quadratic surds.

§1. A FRUSTRATING QUESTION

Let a and b be two coprime integers, $1 < b < a$. *Is it true that the sequence* $(a/b)^n, n = 0, 1, 2, \cdots$ *is dense* (mod 1)? This very old problem of Pisot and Vijayaraghavan is still unanswered. Pisot, Vijayaraghavan and André Weil did however show that there exist infinitely many cluster points.

Are any one of these cluster points irrational? Even this seems unanswered. We address a simpler question, but before we must define the depth of a rational number x: it is simply the length $\delta(x)$ of the continued fraction of x

$$x = [c_0, c_1, c_2, \cdots, c_\delta]$$

where we choose δ to be even ($c_\delta \geqslant 1$). For example

$$\delta(k) = 0,\ k \in \mathbf{Z};\ \delta\left(\frac{1}{2}\right) = 2;\ \delta\left(\frac{3}{5}\right) = 4\ .$$

Quite obviously $\delta(a/b) = O(\ln(b))$, $1 \leqslant b < a$ (see [8]).

Suppose that the sequence $(a/b)^n$ has an irrational cluster point ζ (mod 1). Then some subsequence $(a/b)^{n_j}$ (mod 1) tends to ζ hence

$$\delta((a/b)^{n_j}) \to \infty\ .$$

KEY WORDS: Continued fractions, depth, Pisot numbers, Möbius transformations, quadratic surds.

This observation led me to ask in 1971 [10] whether it was indeed true that

$$\sup_n \delta((a/b)^n) = \infty$$

whithout any other assumptions than $1 < b < a$, $(a, b) = 1$. The problem was solved by Y. Pourchet in an unpublished letter he sent me [14] and by G. Choquet in a series of Comptes Rendus à l'Académie des Sciences [2].

THEOREM 1. *If* a *and* b *are two coprime integers* $1 < b < a$ *then*

$$\lim_{n \to \infty} \delta((a/b)^n) = \infty .$$

Choquet's proof involves dynamical systems. He could only show that the "sup" is infinite. Pourchet's proof is number theoretical and uses the Mahler-Ridout theorem which strengthens Roth's famous result on the rational approximations of algebraic numbers.

It is a pity that Pourchet never published his result. Fortunately A. van der Poorten gave some details of the proof in [16].

§2. A QUESTION CONCERNING PISOT NUMBERS

Let $x > 1$ be a real number. Define the set

$$E(x) = \{x^n \pmod 1 \mid n \in \mathbf{N}\} \subset [0, 1]$$

Let $E'(x)$ be the derived set i.e. the set of cluster points of $E(x)$. Define $E^{(n)}(x)$ recursively to be the derived set of $E^{(n-1)}(x)$, $n \geqslant 1$. In [12] Pisot establishes that if x is a real algebraic number larger that 1 such that $E''(x) = \varnothing$ then x is a Pisot number. I ask the following question.

PROBLEM 1. *Is it true that if* $x > 1$ *is algebraic and if for some* $k \in \mathbf{N}$ $(k \geqslant 2)$ $E^{(k)}(x) = \varnothing$ *then* x *is a Pisot number?*

A positive answer to this problem implies the weak form of Theorem 1, namely that the sup is infinite. Indeed, define $A_0 = \{0\}$ and for $k \geqslant 1$

$$A_{2k} = \{\zeta \in (0, 1) \mid \delta(\zeta) \leqslant 2k\} .$$

Let $A'_{2k}, A''_{2k}, \cdots$ be the derived sets of A_{2k}. Clearly $A'_{2k} = A_{2k-2}$, therefore

$$A_{2k}^{(k+1)} = \varnothing .$$

Now let $x > 1$ be a rational number which is not an integer. Suppose

$$\sup_n \delta(x^n) < \infty$$

Then for some k

$$E(x) \subset A_{2k}$$

hence

$$E^{(k+1)}(x) = \varnothing .$$

Assuming a positive answer to Problem 1, we conclude that x is a Pisot number, i.e. a rational integer. This contradicts the assumption hence

$$\sup_n \delta(x^n) = \infty . \qquad QED$$

§3. More questions on $\delta(x^n)$

H. Heilbronn [7], T. Tonkov [15] and finally J.W. Porter [13] improving on one another established that as a tends to infinity

$$\frac{1}{\varphi(a)} \sum_{\substack{b<a \\ (a,b)=1}} \delta\left(\frac{a}{b}\right) = \frac{12}{\pi^2} \ln 2 \ln a + O(1) .$$

Independtly, J.D. Dixon [6] showed that for all $\varepsilon > 0$ and for all $a, b, 1 < b < a < x$ with the exception of at most $o(x^2)$ couples, one has

$$\left| \delta\left(\frac{a}{b}\right) - \frac{12}{\pi^2} \ln 2 \ln a \right| \leqslant (\ln a)^{\frac{1}{2}+\varepsilon} .$$

See H. Daudé's work for a dual result [5]. These results suggest the second problem.

Problem 2. *Is it true that for all coprime* a *and* b, $1 < b < a$

$$\lim_{n \to \infty} \frac{1}{n} \delta\left(\left(\frac{a}{b}\right)^n\right) = \frac{12}{\pi^2} \ln 2 \ln b \ ? \tag{1}$$

The limit should indeed be what is stated above and not

$$\frac{12}{\pi^2} \ln 2 \ln a .$$

since it is b, the smallest of the two integers a and b that seems to control the behavior of $(a/b)^n$. It is also to be noticed that in Dixon's inequality, the $\ln a$ that appears on the left hand side could well be replaced by $\ln b$ since for "almost all" couples $a, b, \ln a \approx \ln b$.

Numerical evidence supports equality (1). Based on computer computation, Chr. Batut and M. Olivier showed that

$$\left| \frac{1}{n} \delta\left(\left(\frac{a}{b}\right)^n\right) - \frac{12}{\pi^2} \ln 2 \ln b \right|$$

is less than .02 for n in the range (4000, 5000) and for $a = 3$, $b = 2$ on the one hand and $a = 5$, $b = 2$ on the other hand.

§4. Related problems

My initial (unsuccessful) attempts to prove Theorem 1 were based on the comparison of $\delta(ax/b)$ to $\delta(x)$. I was hoping that a relationship between both depths would give by induction some results on $\delta(xa^n/b^n)$. This turned out nonconclusive, yet I did obtain some results which I believe are interesting in themselves [10].

Let a, b, c, d be coprime integers and let $\Delta = |ad - bc|$. Consider the Möbius map

$$x \mapsto Tx = \frac{ax + b}{cx + d}.$$

THEOREM 2.

$$\limsup_{\delta(x) \to \infty} \frac{\delta(Tx)}{\delta(x)} = \Theta(\Delta)$$

where Θ *takes odd integral values. As* n *increases to infinity* $\Theta(n)$ *behaver like* $\ln n$. *More precisely let*

$$\alpha = \left(2 \ln \frac{1 + \sqrt{5}}{2}\right)^{-1}.$$

Then for all integer $n \geqslant 1$

$$1 + \alpha \ln n \leqslant \Theta(n) \leqslant 2(1 + \alpha \ln n).$$

Θ *is linked to the depth by the formula*

$$\Theta(n) = \max_{1 \leqslant b \leqslant n} \delta\left(\frac{b}{n}\right) + 1.$$

Finally

$$\liminf_{\delta(x)\to\infty} \frac{\delta(Tx)}{\delta(x)} = \frac{1}{\Theta(\Delta)} .$$

The following table gives the first values of Θ

n	1	2	3	4	5	6	7	8	9	10	...
$\Theta(n)$	1	3	3	3	5	3	5	5	5	5	...

Actually Theorem 1 can be improved. There exist two constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that for all rational x

$$\frac{1}{\Theta(\Delta)} \delta(x) - C_1 \leqslant \delta(Tx) \leqslant \Theta(\Delta)\,\delta(x) + C_2 .$$

Both inequalities are sharp apart from the exact values of C_1 and C_2.

5. More questions

To every Möbius map T we associate the interval $I(T) = [\Theta^{-1}(\Delta), \Theta(\Delta)]$.

Problem 3. *Is it true that for all* $\zeta \in I(T)$ *there exists a sequence of rational numbers* x_n *such that* $\lim_{n\to\infty} \delta(x_n) = \infty$ *and*

$$\lim_{n\to\infty} \frac{\delta(Tx_n)}{\delta(x_n)} = \zeta ?$$

Problem 4. *Let* $T_1, T_2, \cdots, T_k$ *be Möbius maps with pairwise coprime determinants* $\Delta_1, \Delta_2, \cdots, \Delta_k$.

Is it true that for all

$$(\zeta_1, \zeta_2, \cdots, \zeta_k) \in \prod_{i=1}^{k} I(T_i)$$

there exists a sequence of rational x_n *with strictly increasing depths such that for all* $i = 1, 2, \cdots, k$

$$\lim_{n\to\infty} \frac{\delta(T_i x_n)}{\delta(x_n)} = \zeta_i ?$$

Can k *be infinite?*

The following result should be mentioned at this point.

THEOREM 3. *There exists a sequence of rational numbers* x_n *with strictly increasing depths such that for all Möbius maps* T

$$\lim_{n \to \infty} \frac{\delta(Tx_n)}{\delta(x_n)} = 1 .$$

The proof is quite simple. To each irrational

$$x = [c_0, c_1, c_2, \cdots]$$

we associate the sequence of best approximations

$$x_n = \frac{p_n}{q_n} = [c_0, c_1, c_2, \cdots, c_n] .$$

Paul Lévy [9] showed that for almost all x

$$\ln q_n \sim \frac{\pi^2}{12 \ln 2} n$$

as n goes to infinity (see for example [1] p. 45). In other words, for almost all x

$$\delta(x_n) \sim \frac{12 \ln 2}{\pi^2} \ln q_n .$$

Therefore, for almost all x

$$\delta\left(\frac{ax_n + b}{cx_n + d}\right) \sim \frac{12 \ln 2}{\pi^2} \ln(cp_n + dq_n) .$$

Now $p_n \sim xq_n$ so that

$$cp_n + dq_n \sim (cx + d) q_n$$

$$\ln(cp_n + dq_n) \sim \ln q_n .$$

Hence for almost all x

$$\delta\left(\frac{ax_n + b}{cx_n + d}\right) \sim \delta(x_n) .$$

By countable intersection, we conclude that for almost all x and for all Möbius map T

$$\delta(Tx_n) \sim \delta(x_n) . \qquad \text{QED}$$

PROBLEM 5. *Let* T *be a given Möbius map and let* $I(T)$ *be the associated interval. Let* $\zeta \in I(T)$. *To compute the Hausdorff dimension of those* x *for which*

$$\lim_{n \to \infty} \frac{\delta(Tx_n)}{n} = \zeta .$$

Extend this problem to higher dimensions in the spirit of problem 4.

§6. QUADRATIC SURDS

Let x be a real quadratic number. Its continued fraction expansion is ultimately periodic. Let $\pi(x)$ be its period. H. Cohen [3], followed by J. Cusick [4] and Paysant-Leroux [11] studied the action of a Möbius map on the period. They established that

$$\lim_{\pi(x) \to \infty} \sup \frac{\pi(Tx)}{\pi(x)} = R(\Delta)$$

where $R(\Delta)$ is an integer. Furthermore

$$A\, n \ln n \leqslant R(n) \leqslant B\, n \ln n + 1$$

for some constants $A > 0$, $B > 0$. A simple argument then shows that

$$\lim_{\pi(x) \to \infty} \inf \frac{\pi(Tx)}{\pi(x)} = \frac{1}{R(\Delta)} .$$

PROBLEM 6. *Is it true that for all real quadratic irrational* x

$$\sup_n \pi(x^n) = \infty \ ?$$

Define the interval

$$J(\Delta) = \left[\frac{1}{R(\Delta)}, R(\Delta)\right] .$$

PROBLEM 7. *Let* $\zeta \in J(\Delta)$. *Prove the existence of a sequence of real quadratic numbers* x_n *with strictly increasing period such that*

$$\lim_{n \to \infty} \frac{\pi(Tx_n)}{\pi(x_n)} = \zeta .$$

Extend this result to higher dimensions as in Problem 4.

PROBLEM 8. *Does there exist a sequence* x_n *of quadratic numbers with strictly increasing period such that for all Möbius map* T

$$\lim_{n \to \infty} \frac{\pi(Tx_n)}{\pi(x_n)} = 1 ?$$

We believe some of our problems are relatively easy to solve. But quite obviously Problem 1, 2 and maybe 6 are deep.

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ADDED IN PROOF

1. In a delightful article to appear "Origins of the Analysis of Algorithms", J. Shallit discusses the early history of $\delta(x), x \in \mathbf{Q}$.

2. Recently (summer 1993), G. Grisel (University of Caen, France) managed to show that for a large class of quadratic irrationals $x, \pi(x^n)$ is indeed unbounded so that Problem 6 is partially solved.

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