4. The use of the Multiplier Theorem

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The remaining candidates are listed below, together with an indication in parenthesis showing that each one (except 505) is excluded by Theorem 2 in Section 2: if N has a prime factor p such that $p^f \equiv -1 \bmod N'$, where N' is the largest divisor of N relatively prime to p, then there is no (periodic) Barker sequence of length $4N^2$.

REMAINING CANDIDATES (excluded by Theorem 2, except $N = 505$.)

N		N	
$65 = 5 \cdot 13$	$(5^2 \equiv -1 \bmod 13)$	$425 = 5^2 \cdot 17$	$(5^8 \equiv -1 \bmod 17)$
$85 = 5 \cdot 17$	$(17^2 \equiv -1 \bmod 5)$	$445 = 5 \cdot 89$	$(89 \equiv -1 \bmod 5)$
$145 = 5 \cdot 29$	$(29 \equiv -1 \bmod 5)$	$481 = 13 \cdot 37$	$(37^6 \equiv -1 \bmod 13)$
$185 = 5 \cdot 37$	$(37^2 \equiv -1 \bmod 5)$	$485 = 5 \cdot 97$	$(97^2 \equiv -1 \bmod 5)$
$205 = 5 \cdot 41$	$(5^{10} \equiv -1 \bmod 41)$	$493 = 17 \cdot 29$	$(17^2 \equiv -1 \bmod 29)$
$221 = 13 \cdot 17$	$(13^2 \equiv -1 \bmod 17)$	$505 = 5 \cdot 101$	
$265 = 5 \cdot 53$	$(53^2 \equiv -1 \bmod 5)$	$533 = 13 \cdot 43$	$(43^3 \equiv -1 \bmod 13)$
$305 = 5 \cdot 61$	$(5^{15} \equiv -1 \bmod 61)$	$545 = 5 \cdot 109$	$(109 \equiv -1 \bmod 5)$
$325 = 5^2 \cdot 13$	$(5^2 \equiv -1 \bmod 13)$	$565 = 5 \cdot 113$	$(113^2 \equiv -1 \bmod 5)$
$365 = 5 \cdot 73$	$(73^2 \equiv -1 \bmod 5)$	$629 = 17 \cdot 37$	$(37^8 \equiv -1 \bmod 17)$
$377 = 13 \cdot 29$	$(13^7 \equiv -1 \bmod 29)$	$685 = 5 \cdot 137$	$(137^2 \equiv -1 \bmod 5)$

The case $N = 505 = 5 \cdot 101$ cannot be excluded by Theorem 2, because $101 \equiv 1 \bmod 5$ and $5^{25} \equiv 1 \bmod 101$. However, 505 can still be excluded by Turyn's Inequality, as observed in [JL]: choosing $p = 101$ and $w = 2 \cdot 101^2$, so that p is trivially semi-primitive modulo w, we would have

$$p \leqslant \frac{v}{w} = 2 \cdot 5^2 = 50 \,,$$

a contradiction to the assumed existence of a Barker sequence of length $4 \cdot 505^2$.

The first open case is thus $N = 689 = 13 \cdot 53$. We have $53 \equiv 1 \bmod 13$ and $13^{13} \equiv 1 \bmod 53$, so that neither 53 is semi-primitive mod 13, nor 13 is semi-primitive mod 53. The next open case is $N = 793 = 13 \cdot 61$.

4. THE USE OF THE MULTIPLIER THEOREM

In this section we give the details of some (typical) non-existence proofs needed to establish the tables, using the multiplier theorem.

Recall that if D is a cyclic difference set with parameters (v, k, λ), and if $n = k - \lambda$ is greater than λ, then the group of multipliers of D contains the intersection M in $(\mathbf{Z}/v\mathbf{Z})^*$ of the subgroups generated by $l_1, \ldots, l_r$, where $l_1, \ldots, l_r$ are the prime factors of n.

(1) *Parameters* ($v = 181, k = 81, \lambda = 36$), *Table I with* $t = 9$.

Here, $n = 3^2 \cdot 5$, and since $5 \equiv 3^6 \bmod 181$, the multiplier theorem says that if an abelian difference set exists with these parameters, then 5 is a multiplier. The orbits of the multiplication by 5 in $\mathbf{Z}/181\mathbf{Z}$ are $\{0\}$ and 12 orbits of cardinality 15, e.g.

$$\{1, 5, 25, 125, 82, 48, 59, 114, 27, 135, 132, 117, 42, 29, 145\} .$$

(Note that 181 is a prime number.) No subset of $G = \mathbf{Z}/181\mathbf{Z}$ of cardinality $k = 81$ may thus be a union of orbits.

(2) *Parameters* ($v = 4901, k = 2401, \lambda = 1176$), *Table I with* $t = 49$.

Here, $n = 5^2 \cdot 7^2$. We have $25 = 5^2 \equiv 7^6 \bmod 4901$. Therefore, if an abelian difference set exists, $m = 25$ must be a multiplier. Writing the group $G = \mathbf{Z}/4901\mathbf{Z}$ as $G = \mathbf{Z}/13^2\mathbf{Z} \times \mathbf{Z}/29\mathbf{Z}$, with group operation $(a, b) \cdot (a', b') = (a + a', b + b')$, the orbits under multiplication by $m = 25$ are

$$E = \{(0,0)\}$$

$$U_i = \{(13i, 0), (-13i, 0)\} \quad i = 1, 2, 3, 4, 5, 6$$

$$V_j = \{(j,0), (25j,0), (118j,0), (77j,0), (66j,0), (129j,0), (14j,0), (12j,0), (131j,0), (64j,0), (79j,0), (116j,0), (27j,0), (-j,0), \dots\}$$
$j = 1, \dots, 6$, each V_j of cardinality 26.

$$X = \{(0,1), (0,25), (0,16), (0,23), (0,24), (0,20), (0,7)\}$$

$$Y = \{(0,2), (0,21), (0,3), (0,17), (0,19), (0,11), (0,14)\}$$

$$\bar{X} = \{(0, -x) \mid (0,x) \in X\}$$

$$\bar{Y} = \{(0, -y) \mid (0,y) \in Y\}$$

each of cardinality 7.

There are moreover, the 24 orbits $U_i \cdot X$, $U_i \cdot \bar{X}$, $U_i \cdot Y$, $U_i \cdot \bar{Y}$ of cardinality 14, where

$$A \cdot B = \{a \cdot b \mid a \in A, b \in B\} .$$

Finally, there are 24 orbits $V_i \cdot X$, $V_i \cdot \bar{X}$, $V_i \cdot Y$, $V_i \cdot \bar{Y}$ of cardinality 182. Contrary to the preceding example, there are many ways of writing the cardinality 2401 of a putative difference set D as a sum of numbers taken from the set of orbit cardinalities.

To ease calculations, we view a subset $S \subset G$ as the element $\sum_{s \in S} s$ in the integral group ring. Note that, with this convention, the product $S \cdot T$ in $\mathbf{Z}G$ coincides with the element of $\mathbf{Z}G$ associated with the product set

$S \cdot T = \{s \cdot t \mid s \in S, t \in T\}$. A difference set D, if it exists with the above parameters, can be written as

$$D = C + AX + BY + P\bar{X} + Q\bar{Y}$$

where C, as well as A, B, P, Q, is of the form

$$C = \alpha E + \sum_{i=1}^{6} \beta_i U_i + \sum_{j=1}^{6} \gamma_j V_j$$

with coefficients $\alpha, \beta_1, \ldots, \beta_6, \gamma_1, \ldots, \gamma_6$ all equal to 0 or 1.

As in Section 1, D is a difference set if and only if

$$D\bar{D} = 1225 + 1176 \cdot \left(1 + \sum_{i=1}^{6} U_i + \sum_{j=1}^{6} V_j\right) \cdot (1 + X + \bar{X} + Y + \bar{Y}) .$$

Now, writing $G = G_1 \times G_2$ as above, $G_1 = \mathbf{Z}/13^2\mathbf{Z}$, $G_2 = \mathbf{Z}/29\mathbf{Z}$, let $\pi : \mathbf{Z}G \to \mathbf{Z}G_1$ be the projection on the group ring of G_1. We have $\pi X = \pi\bar{X} = \pi Y = \pi\bar{Y} = 7$, and reducing modulo 7,

$$\pi(D\bar{D}) = C\bar{C} = 0 \text{ in } \mathbf{F}_7 G_1 .$$

The involution of $\mathbf{Z}G$, sending (a, b) to $\overline{(a, b)} = (-a, -b)$, is the identity on U_i, V_j:

$$\bar{U}_i = U_i, \qquad \bar{V}_j = V_j .$$

Therefore $\bar{C} = C$ and $C^2 = 0$ in $\mathbf{F}_7 G_1$. However, $\mathbf{F}_7 G_1$, where G_1 is of order 13^2, prime to 7, is a semi-simple algebra and does not contain any nilpotent element. It follows that $C = 0$ in $\mathbf{F}_7 G_1$. Since the coefficients of $C = \alpha E + \sum_{i=1}^{6} \beta_i U_i + \sum_{j=1}^{6} \gamma_i V_j$ are all 0 or 1, this implies $C = 0$ in $\mathbf{Z}G_1$, i.e.

$$D = AX + BY + P\bar{X} + Q\bar{Y} ,$$

and $\pi D = 7 \cdot S$ with

$$S = r + \sum_{i=1}^{6} s_i U_i + \sum_{j=1}^{6} t_j V_j ,$$

where $S = A + B + P + Q$. Thus, all coefficients $r, s_1, \ldots, s_6, t_1, \ldots, t_6$ are non-negative integers $\leqslant 4$.

Again $\pi(D\bar{D}) = 1225 + 1176 \cdot (1 + \sum U_i + \sum V_j) \cdot 29$. Therefore,

$$S^2 = 25 + 696 \cdot \left(1 + \sum_{i=1}^{6} U_i + \sum_{j=1}^{6} V_j\right) .$$

With our (abuse of) notation, we set $G_1 = 1 + \sum U_i + \sum V_j$. Then, $G_1^2 = 169 \cdot G_1$. Thus, we see that

$$S = \pm(5 + 2G_1)$$

are solutions of $S^2 = 25 + 696 \cdot G_1$. We claim that there is no other. This will clearly finish the non-existence proof since $r \leqslant 4$. Note the decomposition

$$\mathbf{Q}G_1 = \mathbf{Q} \times \mathbf{Q}(\zeta_{13}) \times \mathbf{Q}(\zeta_{169})$$

of the algebra $\mathbf{Q}G_1$ as a product of fields, where ζ_{13} is a primitive 13-th root of unity, and ζ_{169} a primitive 169-th root of unity.

The element $G_1 = \sum_{k=0}^{168} z^k \in \mathbf{Z}G_1$ corresponds on the right hand side to $(169, 0, 0)$ since ζ_{13} and ζ_{169} are roots of the polynomial $\sum_{k=0}^{168} X^k$. It follows that $S^2 = (343^2, 5^2, 5^2)$. Hence, any solution $Z \in \mathbf{Z}G_1$ of the equation $Z^2 = 25 + 696G_1$ must correspond to $(\pm 343, \pm 5, \pm 5)$. Changing Z to $-Z$, we can assume $Z = (343, \pm 5, \pm 5)$. Now, the diagrams

$$\begin{array}{ccc} \mathbf{Z}G_1 & \to & \mathbf{Z}[\zeta_{13}] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \to & \mathbf{F}_{13} \end{array}$$

and

$$\begin{array}{ccc} \mathbf{Z}G_1 & \to & \mathbf{Z}[\zeta_{169}] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \to & \mathbf{F}_{13} \end{array}$$

where the right vertical arrows send ζ_{13}, resp. ζ_{169} to $1 \in \mathbf{F}_{13}$, are commutative. Since 5 is not congruent to -5 modulo 13, and 343 maps to $+5 \in \mathbf{F}_{13}$, we see that $Z = (343, 5, 5) = S$.

(3) *Parameters* ($v = 13613$, $k = 6724$, $\lambda = 3321$), *Table I with* $t = 82$.

This case is as simple as case (1). Indeed, $n = 3403 = 41 \cdot 83$. Since $41 \equiv 83^3 \bmod 13613$, it follows from the multiplier theorem that if a cyclic difference set D with parameters (13613, 6724, 3321) existed, then 41 would be a multiplier, and D could be taken to be a union of orbits under multiplication by 41 on the cyclic group $\mathbf{Z}/13613\mathbf{Z}$.

The order of 41 modulo 13613 is 3403, and beside the one-point orbit $\{0\}$, there are 4 orbits X, iX, i^2X, i^3X each of cardinality 3403, where

$$X = \{1, 41, 1681, \ldots, 13281\}$$

and i is a square root of -1 mod 13613, e.g. $i = 165$. Note that 13613 is a prime number.

However, 6724 is not of the form $n_0 + 3403n_1$ with $n_0 = 0$ or 1 and $0 \leqslant n_1 \leqslant 4$. No difference set can therefore have the above parameters.

(4), (5), (6) *Parameters* $(v,k,\lambda) = (3^3,13,6)$, $(3^5,121,60)$ *and* $(7^3,171,85)$ *of Table II, with* $n = 7,61$ *and* 86 *respectively.*

More generally, we will consider the case

$$(v,k,\lambda) = \left(p^{2t+1}, \frac{p^{2t+1}-1}{2}, \frac{p^{2t+1}-3}{4}\right),$$

where p is a prime $\equiv 3$ mod 4.

We have $n = k - \lambda = \dfrac{p^{2t+1}+1}{4}$. Let $l_1, \ldots, l_r$ be the primes dividing n. The group of multipliers for a putative difference set D with the above parameters contains the intersection M in $(\mathbf{Z}/v\mathbf{Z})^*$ of the subgroups generated by $l_1, \ldots, l_r$. Since $(\mathbf{Z}/v\mathbf{Z})^*$ is cyclic, M is the unique subgroup of $(\mathbf{Z}/v\mathbf{Z})^*$ whose order is the greatest common divisor of the orders $q_1, \ldots, q_r$ of $l_1, \ldots, l_r$ in $(\mathbf{Z}/v\mathbf{Z})^*$. We will now assume that the orders $q_1, \ldots, q_r$ of the prime factors $l_1, \ldots, l_r$ of $n = k - \lambda$ in $(\mathbf{Z}/v\mathbf{Z})^*$ are all divisible by p^{t+1}.

THEOREM. *There is no cyclic difference set with parameters*

$$(v,k,\lambda) = \left(p^{2t+1}, \frac{p^{2t+1}-1}{2}, \frac{p^{2t+1}-3}{4}\right),$$

where p *is a prime* $\equiv 3$ mod 4, *provided that the orders* $q_1, \ldots, q_r$ *of the prime factors* $l_1, \ldots, l_r$ *of* $n = k - \lambda$ *in* $(\mathbf{Z}/v\mathbf{Z})^*$ *are all divisible by* p^{t+1}.

Note that the hypotheses of the theorem above are satisfied for the three examples we have in mind. (Cases $n = 7, 61$ and 86 in Table II.)

(1) $n = 7: p = 3, t = 1$, and 7 is of order 3^2 modulo 27;

(2) $n = 61: p = 3, t = 2$, and 61 is of order 3^4 modulo 243;

(3) $n = 86: p = 7, t = 1$, and 2 is of order $3 \cdot 7^2$ modulo 343, 43 is of order 7^2 modulo 343.

As expected, the hypothesis on the orders of the prime factors of n is not satisfied in general. It fails for instance for $p = 11$, $t = 1$: here $n = \dfrac{11^3+1}{4} = 333 = 3^2 \cdot 37$ and whereas 37 is of order $5 \cdot 11^2$ modulo 11^3, 3 is only of order $5 \cdot 11$ modulo 11^3.

However, failure of the hypothesis seems fairly rare: the next example with $t = 1$ occurs for $p = 3511$. Note that 3511 is special for another reason: it satisfies the congruence $2^{p-1} \equiv 1 \bmod p^2$, the only other known solution being the famous $p = 1093$. Such prime numbers are known in the literature as Wieferich prime numbers.

The behaviour of the orders of the prime factors of $n = \dfrac{p^{2t+1}+1}{4}$ in $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$ is probably a difficult question.

Proof of the Theorem. The hypothesis on the orders $q_1, \ldots, q_r$ means that $m = 1 + p^t$, which generates the subgroup of order p^{t+1} in $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$, is contained in all the subgroups $< l_1 >, \ldots, < l_r >$ of $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$, and thus is a multiplier of any candidate difference set $D \subset \mathbf{Z}/p^{2t+1}\mathbf{Z}$ with the above parameters.

What are the orbits of multiplication by $m = 1 + p^t$ in the ring $\mathbf{Z}/p^{2t+1}\mathbf{Z}$? If $a_i = i \cdot p^{t+1}$, then $a \cdot m \equiv a \bmod p^{2t+1}$. Hence, there are p^t fixed points $a_0 = 0, a_1, \ldots, a_{p^t-1}$.

More generally, if $a_{i,j} = ip^{t-j+1}$ with $1 \leqslant i \leqslant p^t - 1$ and $\gcd(i,p) = 1$, $j = 1, \ldots, t+1$, then $a_{i,j}$ produces an orbit $\{a_{i,j}m^\nu\}_{\nu = 0, \ldots, p^j - 1}$ of length p^j. Here, we use the formula

$$(1 + p^t)^{p^s} \equiv 1 + p^{t+s} \mod (p^{t+s+1})$$

easily proved (for p odd) by induction on s, and which implies that m has (multiplicative) order p^j modulo p^{t+j}.

The orbits $A_{i,j}$ of $a_{i,j}$ with $i \in \mathbf{Z}/p^t\mathbf{Z}$ for $j = 0$ $(a_{i,0} = a_i)$, and $i \in (\mathbf{Z}/p^t\mathbf{Z})^*$ for $j = 1, \ldots, t+1$ are easily verified to be disjoint. Together, they sweep out

$$p^t + \sum_{j=1}^{t+1} (p-1)p^{t-1}\, p^j = p^{2t+1}$$

elements of the group $\mathbf{Z}/p^{2t+1}\mathbf{Z}$. Hence, $A_{i,j}$ with $i \in \mathbf{Z}/p^t\mathbf{Z}$ for $j = 0$ $(a_{i,0} = a_i)$, and $i \in (\mathbf{Z}/p^t\mathbf{Z})^*$ for $j = 1, \ldots, t+1$ is the complete collection of orbits under multiplication by $m = 1 + p^t$ in $\mathbf{Z}/p^{2t+1}\mathbf{Z}$. At this point, it may be more convenient to write the group ring of $\mathbf{Z}/p^{2t+1}\mathbf{Z}$ as $\mathbf{Z}[x]/(x^{p^{2t+1}} - 1)$. Identifying a subset $A \subset \mathbf{Z}/p^{2t+1}\mathbf{Z}$ with the sum of the corresponding elements $\sum_{a \in A} a$ in the group ring, the orbits $A_{i,j}$ can then be written as

$$A_{i,j} = \sum_{\nu=0}^{p^j-1} x^{ip^{t-j+1}m^\nu}.$$

If a difference set D with the above parameters exists, it must be of the form

$$D = \sum_{i \in S_0} x^{ip^{t+1}} + \sum_{j=1}^{t+1} \sum_{i \in S_j} A_{i,j}$$

where $S_0 \subset \mathbf{Z}/p^t\mathbf{Z}$ and $S_j \subset (\mathbf{Z}/p^t\mathbf{Z})^*$ for $j = 1, \ldots, t+1$. Now, let $\pi : \mathbf{Z}[x]/(x^{p^{2t+1}} - 1) \to \mathbf{Z}[y]/(y^p - 1)$ be the projection of the group ring of $\mathbf{Z}/p^{2t+1}\mathbf{Z}$ onto the group ring of the cyclic group of order p. We have $\pi(x) = y$ and

$$\pi A_{i,j} = p^i \quad \text{for} \quad j = 0, 1, \ldots, t$$

$$\pi A_{i,t+1} = p^{t+1} \cdot y^i \quad \text{for} \quad i \in (\mathbf{Z}/p^t\mathbf{Z})^* .$$

It follows that

$$\pi D = s_0 + ps_1 + \cdots + p^t s_t + p^{t+1} \left(\sum_{i \in S_{t+1}} y^i \right),$$

where $s_j = \mathrm{Card}(S_j)$.

Let $N = s_0 + ps_1 + \cdots + p^t s_t$ and $a_\mu = \mathrm{Card}\{i \mid i \in S_{t+1}, i \equiv \mu \bmod p\}$, then

$$\pi D = N + p^{t+1} Y,$$

with $Y = \sum_{\mu=1}^{p-1} a_\mu y^\mu$. (Note that a_0 is indeed 0 as $S_{t+1} \subset (\mathbf{Z}/p^t\mathbf{Z})^*$.) Therefore $\pi(D\bar{D}) = \pi(D)\overline{\pi(D)}$ has the form

$$\pi(D\bar{D}) = N^2 + Np^{t+1} \sum_{\mu=1}^{p-1} a_\mu (y^\mu + y^{-\mu}) + p^{2t+2} Y\bar{Y} .$$

On the other hand the condition for D being a difference set yields, after applying π,

$$\pi(D\bar{D}) = \frac{p^{2t+1}+1}{4} + \frac{p^{2t+1}-3}{4} p^{2t} \left(\sum_{\mu=0}^{p-1} y^\mu \right) .$$

We will reach a contradiction by comparing the constant terms (coefficient of 1 in $\mathbf{Z}[y]/(y^p - 1)$) in the two expressions for $\pi(D\bar{D})$:

$$N^2 + p^{2t+2} \sum_{\mu=1}^{p-1} a_\mu^2 = \frac{p^{2t+1}+1}{4} + \frac{p^{2t+1}-3}{4} p^{2t} .$$

Note that $k = \mathrm{Card}(D) = N + p^{t+1} s_{t+1}$, where $s_{t+1} = \mathrm{Card}(S_{t+1})$, and hence $N = \dfrac{p^{2t+1}-1}{2} - p^{t+1} s_{t+1}$. Substituting this in the above equation, we get

$$4s_{t+1} \equiv 3p^{t-1}(p-1) \mod p^{t+1} .$$

Writing $4s_{t+1} = 3p^{t-1}(p-1) + z \cdot p^{t+1}$ for $z \in \mathbf{Z}$, we observe that $p \equiv 3 \mod 4$ implies $z \equiv 2 \mod 4$, and so $2p^{t+1} \leqslant |z \cdot p^{t+1}|$. But, $s_{t+1} = \mathrm{Card}(S_{t+1}) \leqslant p^{t-1}(p-1)$, since $S_{t+1} \subset (\mathbf{Z}/p^t\mathbf{Z})^*$. It follows that

$$|z \cdot p^{t+1}| \leqslant |4s_{t+1} - 3p^{t-1}(p-1)| \leqslant 3p^{t-1}(p-1) < 2p^{t+1} \leqslant |z \cdot p^{t+1}| .$$

We have reached the desired contradiction, i.e. no cyclic difference set with parameters $\left(p^{2t+1}, \dfrac{p^{2t+1}-1}{2}, \dfrac{p^{2t+1}-3}{4}\right)$ exists if the orders of the prime factors of $n = \dfrac{p^{2t+1}+1}{4}$ in $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$ are all divisible by p^{t+1}. □

(7) *Parameters* ($v = 399, k = 199, \lambda = 99$), *Table II.* This is the last item in Table II, corresponding to $n = k - \lambda = 100$.

Since $4 = 2^2 \equiv 5^8 \mod 399$, it follows that 4 must be a multiplier of any abelian difference set D with the above parameters.

Writing $\mathbf{Z}/399\mathbf{Z}$ as a direct product

$$\mathbf{Z}/399\mathbf{Z} = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/19\mathbf{Z} ,$$

and accordingly writing the elements of $\mathbf{Z}/399\mathbf{Z}$ as triples $g = (x, y, z)$, $x \in \mathbf{Z}/3\mathbf{Z}$, $y \in \mathbf{Z}/7\mathbf{Z}$, $z \in \mathbf{Z}/19\mathbf{Z}$, we have the following orbits of the multiplication by 4 in $\mathbf{Z}/399\mathbf{Z}$: all monomials XYZ, with $X \in \{1, U, \bar{U}\}$, $Y \in \{1, V, \bar{V}\}$, $Z \in \{1, W, \bar{W}\}$, where

$$1 = \{(0,0,0)\}$$

$$U = \{(1,0,0)\}$$

$$V = \{(0,1,0), (0,-3,0), (0,2,0)\}$$

$$W = \{(0,0,1), (0,0,4), (0,0,-3), (0,0,7), (0,0,9), (0,0,-2), (0,0,-8), (0,0,6), (0,0,5)\} ,$$

and bar denotes the conjugate, i.e. if $C \subset \mathbf{Z}/v\mathbf{Z}$, then $\bar{C} = \{-g \mid g \in C\}$.

All orbits, except $1, U, \bar{U}$ have cardinality divisible by 3. Since $k = 199 \equiv 1 \mod 3$, any putative difference set D can be assumed to contain a single one-point orbit 1, U or $\bar{U}$. Multiplying D by U or $\bar{U}$ if necessary, we may assume that

$$D = 1 + A \cdot V + B \cdot \bar{V} + P \cdot W + Q \cdot \bar{W} ,$$

where

$$A = \alpha_0 + \alpha_1 U + \alpha_2 \bar{U} , \ 0 \leqslant \alpha_i \leqslant 1 ,$$

$$B = \beta_0 + \beta_1 U + \beta_2 \bar{U} , \ 0 \leqslant \beta_i \leqslant 1 ,$$

and P, Q are polynomials in $U, \bar{U}$ and $V, \bar{V}$.

We first show that A and B must be 0. Let $a = \alpha_0 + \alpha_1 + \alpha_2$, $b = \beta_0 + \beta_1 + \beta_2$, and let $\pi: \mathbf{Z}/399\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$ be the projection on the second factor.

We indulge in various abuses of notation: we write π for the group ring projection as well and denote πV again by V. Note that $\pi U = \pi \bar{U} = 1$, $\pi W = \pi \bar{W} = 9$. Then $\pi D \equiv 1 + aV + b\bar{V} \bmod 9$, a congruence in the group ring of $\mathbf{Z}/7\mathbf{Z}$.

Since $D\bar{D} = 100 + 99 \cdot (1 + U + \bar{U})(1 + V + \bar{V})(1 + W + \bar{W})$, the equation expressing that D is a difference set with the required parameters, we have $D\bar{D} \equiv 1 \bmod 9$.

Consequently, using

$$V\bar{V} = 3 + V + \bar{V}\,, \qquad V^2 = V + 2\bar{V}\,, \qquad \bar{V}^2 = 2V + \bar{V}\,,$$

we get, expanding $\pi(D\bar{D}) = \pi(D)\pi(\bar{D})$, and after collecting terms,

$$3(a^2 + b^2) + (a + b + a^2 + b^2 + 3ab)(V + \bar{V}) \equiv 0 \mod 9\,.$$

Thus, $a^2 + b^2 \equiv 0 \bmod 3$, and this means $a \equiv b \equiv 0 \bmod 3$. But then $a^2 + b^2 + 3ab \equiv 0 \bmod 9$, and so we must also have

$$a + b \equiv 0 \mod 9\,,$$

after looking at the coefficient of $V + \bar{V}$ in the above congruence.

Since $0 \leqslant a \leqslant 3, 0 \leqslant b \leqslant 3$, this means $a = b = 0$ and therefore $A = B = 0$. Any difference set D with parameters (399, 199, 99) can therefore be assumed to have the form

$$D = 1 + P \cdot W + Q \cdot \bar{W}\,.$$

Plugging $D = 1 + P \cdot W + Q \cdot \bar{W}$ into the equation

$$D\bar{D} = 100 + 99(1 + U + \bar{U})(1 + V + \bar{V})(1 + W + \bar{W})$$

and using the multiplication table

$$W\bar{W} = 9 + 4(W + \bar{W})\,, \quad W^2 = 4W + 5\bar{W}\,,$$

we get

$$1 + 9(P\bar{P} + Q\bar{Q}) = 100 + 99(1 + U + \bar{U})(1 + V + \bar{V})$$

$$P + \bar{Q} + 4(P\bar{P} + Q\bar{Q}) + 5\bar{P}Q + 4P\bar{Q} = 99(1 + U + \bar{U})(1 + V + \bar{V})\,,$$

where

$$P = p_0 + p_1 U + p_2 \bar{U} + (p_3 + p_4 U + p_5 \bar{U})V + (p_6 + p_7 U + p_8 \bar{U})\bar{V}$$

$$Q = q_0 + q_1 U + q_2 \bar{U} + (q_3 + q_4 U + q_5 \bar{U})V + (q_6 + q_7 U + q_8 \bar{U})\bar{V}$$

with $0 \leqslant p_i, q_i \leqslant 1$, for $i = 0, \ldots, 8$.

The first equation gives

$$P\bar{P} + Q\bar{Q} = 11 + 11(1 + U + \bar{U})(1 + V + \bar{V}) .$$

Substituting in the second equation, we get

$$(*) \qquad P + \bar{Q} + 5\bar{P}Q + 4P\bar{Q} = -44 + 55(1 + U + \bar{U})(1 + V + \bar{V}) .$$

Since $U\bar{U} = 1$, $U^2 = \bar{U}$ and $V\bar{V} = 3 + V + \bar{V}$, $V^2 = V + 2\bar{V}$, the constant terms in $\bar{P}Q$ and $P\bar{Q}$ are equal to $\sum_{i=0}^{2} p_i q_i + 3 \sum_{j=3}^{8} p_j q_j = c$, say. Hence, equating constant terms in the above equation (*), we must have

$$p_0 + q_0 + 9c = 11 .$$

The only solution to this equation with all p_i, q_i being 0 or 1, is $p_0 = q_0 = 1$, $p_i = q_i = 0$ for $i = 1, \dots, 8$. This means $P = Q = 1$, contradicting (*).

5. Comments on the examples in Tables II

Difference sets with parameters $(v, k, \lambda) = (4n - 1, 2n - 1, n - 1)$ are usually called *Hadamard difference sets.* Our purpose here is to discuss the classification of these cyclic difference sets for $2 \leqslant n \leqslant 100$.

In many cases where $v = 4n - 1$ is a prime p, the quadratic residue difference set, which we denote by $QR(p)$ is unique for the given values of the parameters. This is obviously the case if the multiplier m has order $k = \frac{1}{2}(v - 1)$ in $(\mathbf{Z}/v\mathbf{Z})^*$. Indeed, in this case, there are exactly 3 orbits of multiplication by m in $\mathbf{Z}/v\mathbf{Z}$, namely $1 = \{0\}$, $M = \{1, m, m^2, \dots, m^{k-1}\}$ and $\bar{M} = \{-1, -m, \dots, -m^{k-1}\}$. Thus the only choice for D is $D = M$ or $D = \bar{M}$, which are isomorphic under conjugation $\sigma: \mathbf{Z}/v\mathbf{Z} \to \mathbf{Z}/v\mathbf{Z}$, $\sigma(a) = -a$.

In our Table II, this situation happens for $n = 3, 5, 6, 12, 15, 17, 18, 20, 21, 27, 33, 35, 41, 42, 45, 48, 53, 57, 60, 63, 66, 68, 77, 87, 90$ and 96.

The remaining cases where $v = 4n - 1$ is a prime p (for $2 \leqslant n \leqslant 100$) have been shown to lead to a single difference set, namely $QR(p)$, by machine enumeration of the various choices of D as a union of orbits under multiplication by a multiplier m. This includes the cases $n = 26$ (multiplier 8), $n = 38$ (multiplier 19), $n = 50$ (multiplier 5), $n = 78$ (multiplier 13), $n = 83$ (multiplier 83), and $n = 95$ (multiplier 5). By far, the most difficult case (for the machine) occurs with $n = 38$, which required the examination of 37 442 160 combinations of multiplier orbits.