§5. SURJECTIVITY OF $(j_+)_\star$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

is equivalent to the independence of the cohomology classes of these cocycles $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$. To show the independence we use the following theorem.

THEOREM (4.1). *Let* $j_+ : PL_c([0, \infty)) \to \mathbf{R}$ *denote the homomorphism defined by*

$$j_+(f) = \log f'(0) .$$

The homomorphism j_+ *induces a surjection in integer homology.*

Using this theorem, we can show the independence. Let $u_i^- \otimes_{\mathbf{Q}} u_i^+$ be an element of $V^{k_i^-, k_i^+}$ $(u_i^- \in \mathbf{R}^{\wedge k_i^-}, u_i^+ \in \mathbf{R}^{\wedge k_i^+})$. Then we have a k_i^--dimensional cycle σ_i^- of $BPL_c((-\infty, 0])^\delta$ such that the image under $(j_-)_*$ coincides with $u_i^- \in \mathbf{R}^{\wedge k_i^-} \cong H_{k_i^-}(B\mathbf{R}^\delta; \mathbf{Z})$, where $j_- : PL_c((-\infty, 0]) \to \mathbf{R}$ denotes the homomorphism defined by $j_-(f) = \log f'(0)$. We also have a k_i^+-dimensional cycle σ_i^+ of $BPL_c([0, \infty))^\delta$ such that the image under $(j_+)_*$ coincides with $u_i^+ \in \mathbf{R}^{\wedge k_i^+} \cong H_{k_i^+}(B\mathbf{R}^\delta; \mathbf{Z})$. Then $\sigma_i^- \times \sigma_i^+$ is a $(k_i^- + k_i^+)$ -dimensional cycle of $B(PL_c((-\infty, 0]) \times PL_c([0, \infty)))^\delta$ such that the image under $(j_- \times j_+)_*$ coincides with $u_i^- \otimes_{\mathbf{Q}} u_i^+ \in V^{k_i^-, k_i^+}$. Now let $T_1, \ldots, T_s$ be translations of $\mathbf{R}$ such that $T_1(0) < \ldots < T_s(0)$ and the supports of $\sigma_i = T_i(\sigma_i^- \times \sigma_i^+) T_i^{-1}$ are contained in disjoint open intervals, where the support of a cycle of $BPL_c(\mathbf{R})^\delta$ is the union of the supports of the homeomorphisms which appear in the expression of the cycle. Then $\sigma_1 \times \ldots \times \sigma_s$ is an m-cycle and the value of the cocycle $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$ on it is $(u_1^- \otimes_{\mathbf{Q}} u_1^+) \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} (u_s^- \otimes_{\mathbf{Q}} u_s^+)$. It is easy to see that the values of the other m-cocycles on this cycle are 0.

The fact that $*$-product coincides with the tensor product follows from Lemma (1.2). Note that the map s in Lemma (1.2) is an isomorphism from the subgroup of $H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ generated by the $\sigma^- \times \sigma^+$ to $H_{*+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$. Thus Theorem (3.1) is proved.

§5. SURJECTIVITY OF $(j_+)_*$

We prove Theorem (4.1). We consider j_+ as a homomorphism from $PL_c([0, \infty))$ to the group of germs at 0. We use the fact that the n-dimensional homology group of $B\mathbf{R}^\delta$ is isomorphic to $\mathbf{R}^{\wedge n}$ and whose generators are represented by the images of the fundamental classes of tori T^n of dimension n under the mappings which are defined by n (commuting) elements. We will construct an n-complex Y_n with the fundamental class and a degree one map $Y_n \to T^n$. Then for each mapping $T^n \to B\mathbf{R}^\delta$, we will construct a mapping $Y_n \to BPL_c([0,\infty))^\delta$ such that the following diagram commutes.

$$\begin{array}{ccc} Y_n & \to & BPL_c([0,\infty))^\delta \\ \downarrow & & \downarrow \\ T^n & \to & B\mathbf{R}^\delta . \end{array}$$

Theorem (4.1) follows immediately from this commutative diagram.

Construction of Y_n. Let L be a large positive real number. In the Euclidean n space, we consider the following polyhedron X_n

$$X_n = \{(x_1, \dots, x_n) \in [0, L]^n ;\quad x_{i_1} + \dots + x_{i_k} \geqslant (k-1)k/2 \\ \text{for}\quad 1 \leqslant i_1 < \dots < i_k \leqslant n\} .$$

The shape of X_n is the cube with certain neighborhoods of the k-faces ($k \leqslant n-2$) in the coordinate planes deleted, those of the $(k-1)$-faces being thicker than those of the k-faces.

The polyhedron X_n has $2^n - 1 + n$ faces of dimension $n-1$. If $(x_1, \dots, x_n)$ is a vertex of X_n then $(x_1, \dots, x_n)$ is a permutation of $(0, 1, .., k, L, \dots, L)$. In this case we say $(x_1, \dots, x_n)$ is a vertex of type $\{0, 1, .., k, L, \dots, L\}$. There are edges between $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ of the same type $\{0, 1, .., k, L, \dots, L\}$ if one is obtained from the other by permuting two coordinates. The edges between different types exists only if the types are $\{0, 1, .., k-1, L, \dots, L\}$ and $\{0, 1, .., k, L, \dots, L\}$, and one vertex is obtained from the other by changing the entries k and L.

The polyhedron X_n has the $(n-1)$-face $\{x_i = L\}$ which is isometric to X_{n-1}. The $(n-1)$-face $\{x_i = 0\}$ is isometric to X_{n-1} with L replaces by $L - 1$ because if $x_i = 0$ then

$$x_{i_1} + \dots + x_{i_k} \geqslant (k-1)k/2$$

for $\{i_1, \dots, i_k\}$ containing i implies

$$(x_{i_1} - 1) + \dots + (x_{i_k} - 1) \geqslant (k-1)k/2$$

for $\{i_1, \dots, i_k\}$ not containing i. Hence we can define a simplicial identification between the faces $\{x_i = L\}$ and $\{x_i = 0\}$. In general, the face

$$\{x_{i_1} + \dots + x_{i_k} = (k-1)k/2\}$$

is isometric to $X'_{n-k} \times \Sigma_k$, where X'_{n-k} is X_{n-k} with L replaced by $L-k$ and Σ_k is the face $\{x_1 + \dots + x_k = (k-1)k/2\}$ in X_k. The reason is

$$x_{i'_1} + \dots + x_{i'_{k'}} \geqslant (k'-1)k'/2$$

for $\{i'_1, \dots, i'_{k'}\}$ containing $\{i_1, \dots, i_k\}$ implies

$$(x_{i'_1} - k) + \dots + (x_{i'_{k'}} - k) \geqslant (k'-1)k'/2$$

for $\{i'_1, \dots, i'_{k'}\}$ not containing $\{i_1, \dots, i_k\}$. We also fix a simplicial identification between X'_{n-k} and X_{n-k}. Now we distinguish the faces by the set $\{i_1, \dots, i_k\}$ of indices and we see that

$$\partial X_n = \bigcup_{A \subset \{1,\dots,n\},\ \# A \geqslant 2} X_{\{1,\dots,n\}-A} \times \Sigma_A$$
$$\cup \bigcup_i X^{(L)}_{\{1,\dots,n\}-\{i\}} \cup \bigcup_i X^{(0)}_{\{1,\dots,n\}-\{i\}},$$

where

$$X_{\{1,\dots,n\}-A} \times \Sigma_A = \{x_{i_1} + \dots + x_{i_k} = (k-1)k/2\} \quad \text{if} \quad A = \{i_1, \dots, i_k\},$$
$$X^{(L)}_{\{1,\dots,n\}-\{i\}} = \{x_i = L\} \quad \text{and} \quad X^{(0)}_{\{1,\dots,n\}-\{i\}} = \{x_i = 0\}.$$

The complex Y_n is defined inductively as follows. $Y_1 = X_1 = [0, L]$. Y_2 is obtained from X_2 (a pentagon) by identifying $X^{(L)}_{\{i\}}$ and $X^{(0)}_{\{i\}}$ $(i = 1, 2)$ and by taking the double of it. Hence Y_2 is a surface of genus 2. We call the new part in the double $B\Sigma_{\{1,2\}}$.

$$Y_2 = X_2 + B\Sigma_{\{1,2\}}.$$

Y_3 is obtained from X_3 by identifying $X^{(L)}_{\{i,j\}}$ and $X^{(0)}_{\{i,j\}}$ $(i, j = 1, 2, 3)$, by attaching $X_{\{k\}} \times B\Sigma_{\{i,j\}}$ $(\{i,j,k\} = \{1,2,3\})$ to each $X_{\{k\}} \times \Sigma_{\{i,j\}}$, and then by taking the double. The boundary before taking the double is a surface of genus 6. We call the new part in the double $B\Sigma_{\{1,2,3\}}$.

$$Y_3 = X_3 + \sum_{\{i_1,i_2\} \subset \{1,2,3\}} X_{\{1,2,3\}-\{i_1,i_2\}} \times B\Sigma_{\{i_1,i_2\}} + B\Sigma_{\{1,2,3\}}.$$

In general, we define Y_n to be the double of

$$X_n + \sum_{A \subset \{1,\dots,n\},\ \# A \geqslant 2} X_{\{1,\dots,n\}-A} \times B\Sigma_A$$

and we call the new part in the double $B\Sigma_{\{1,\dots,n\}}$.

$$Y_n = X_n + \sum_{A \subset \{1,\dots,n\},\ \# A \geqslant 2} X_{\{1,\dots,n\}-A} \times B\Sigma_A + B\Sigma_{\{1,\dots,n\}}.$$

The mapping from Y_n to T^n is the one which sends the all $B\Sigma_A$ parts to a point and X_n to the fundamental domain of T^n.

Construction of $Y_n \to BPL_c([0,\infty))^\delta$. Now given a mapping $T^n \to B\mathbf{R}^\delta$, we construct a mapping $Y_n \to BPL_c([0,\infty))^\delta$. In other words, given a homomorphism $\mathbf{Z}^n \to \mathbf{R}$, we construct a homomorphism $\pi_1(Y_n) \to PL_c([0,\infty))$. This is also done inductively.

For $n = 1$, it is only necessary to choose a lift in $PL_c([0,\infty))$ of an element of $\mathbf{R}$.

Now for $n = 2$, we choose lifts f_1, f_2 of the generators of $\mathbf{Z}^2$. To the edges of Y_2, we associate elements of $PL_c([0,\infty))$. We put f_1 on the edges of X_2 from (L,L) to $(0,L)$ and from $(L,0)$ to $(1,0)$, and we put f_2 on the edges of X_2 from (L,L) to $(L,0)$ and from $(0,L)$ to $(0,1)$. Then we put the commutator $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$ on the edge from $(0,1)$ to $(1,0)$. Note that the support of this commutator does not contain 0 hence this commutator is an element of $PL_c((0,\infty))$. This commutator is also written as a commutator of elements of $PL_c((0,\infty))$. We can do it very easily, not by using the perfectness of the group $PL_c((0,\infty))$, but by using a conjugation by an element of $PL_c(\mathbf{R})$ which sends 0 to $a(>0)$ and which is the identity on $(2a,\infty)$ when the support of $[f_1, f_2]$ is contained in $(2a,\infty)$. We call this conjugation c_*. (This technique using conjugation is similar to that in [12].) c_* is an isomorphism from $PL_c([0,\infty))$ to a subgroup of $PL_c((0,\infty))$. Then $[f_1, f_2] = c_*([f_1, f_2]) = [c_* f_1, c_* f_2]$ and we associate $c_* f_1, c_* f_2$ to the edges in the new part in the double (in the mirror). Thus we defined the desired mapping $Y_2 \to BPL_c([0,\infty))^\delta$.

For general n, we use the same strategy. First we choose lifts $f_1, \ldots, f_n$ of the generators of $\mathbf{Z}^n$. To the edges of X_n, we associate elements of $PL_c([0,\infty))$. We associate f_i to the edge from a vertex of type $\{0,1,..,k-1,L,\ldots,L\}$ to a vertex of type $\{0,1,..,k,L,\ldots,L\}$ if the i-th coordinate changes from L to k. Then the elements associated to other edges are uniquely determined. In fact, we can associate an element of $PL_c([0,\infty))$ to each vertices as follows. We associate id to the vertex of type $\{L,\ldots,L\}$, if we already associated an element f_v to a vertex v of type $\{0,1,..,k-1,L,\ldots,L\}$ and a vertex v' is obtained from v by changing the i-th coordinate from L to k then we associate $f_i f_v$ to the vertex v'. Thus the edge from one vertex v_1 to another vertex v_2 is associated with $f_{v_2} f_{v_1}^{-1}$. Now if we look at the edges of Σ_A in the $(n-1)$-face $X_{\{1,\ldots,n\}-A} \times \Sigma_A$ the associated elements are in $PL_c((0,\infty))$. By induction, we can find $B\Sigma_A$ with edges in $PL_c((0,\infty))$. Thus we find the boundary of

$$X_n + \sum_{A \subset \{1,\ldots,n\},\ \# A \geqslant 2} X_{\{1,\ldots,n\}-A} \times B\Sigma_A$$

is a cycle of $PL_c((0, \infty))$. Here the products are considered as in the following remark. Hence in the double Y_n, we can associate the images under c_* in the new part of the double. (c_* is the conjugation by an element of $PL_c(\mathbf{R})$ which sends 0 to $a'(>0)$ and which is the identity on $(2a', \infty)$ when the support of the above boundary is contained in $(2a', \infty)$.) Thus we defined the desired mapping $Y_n \to BPL_c([0, \infty))^\delta$. This proves Theorem (4.1).

Remark. For two simplices $(g_1, \ldots, g_m)$ and $(h_{m+1}, \ldots, h_{m+n})$ of the classifying space for a discrete group, we define the product of them as follows.

$$(g_1, \ldots, g_m) \times (h_{m+1}, \ldots, h_{m+n}) = \sum_{\sigma} \operatorname{sign}(\sigma)\, (f_{\sigma,1}, \ldots, f_{\sigma, m+n}) .$$

where the sum is taken over the shuffles σ (that is, those permutations such that $\sigma(1) < \ldots < \sigma(m)$ and $\sigma(m+1) < \ldots < \sigma(m+n)$. The entry $f_{\sigma,j}$ is defined as follows.

$$f_{\sigma,\sigma(j)} = g_j \quad (j = 1, \ldots, m) \quad \text{and}$$
$$f_{\sigma,m+j} = (g_k \cdots g_m) h_{m+j} (g_k \cdots g_m)^{-1} \quad (j = 1, \ldots, n) ,$$

where k is the integer such that $\sigma(k-1) < \sigma(m+j) < \sigma(k)$. For example, $(g_1, g_2) \times (h_3, h_4)$

$$\begin{aligned} &= (g_1, g_2, h_3, h_4) - (g_1, g_2 h_3 g_2^{-1}, g_2, h_4) \\ &\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2, h_4) + (g_1, g_2 h_3 g_2^{-1}, g_2 h_4 g_2^{-1}, g_2) \\ &\quad - (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2 h_4 g_2^{-1}, g_2) \\ &\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1 g_2 h_4 (g_1 g_2)^{-1}, g_1, g_2). \end{aligned}$$

This product is defined so that

$$\begin{aligned} &\partial((g_1, \ldots, g_m) \times (h_{m+1}, \ldots, h_{m+n})) \\ &= (\partial'(g_1, \ldots, g_m)) \times (h_{m+1}, \ldots, h_{m+n}) \\ &\quad + (-1)^m (g_1, \ldots, g_{m-1}) \times (g_m h_{m+1} g_m^{-1}, \ldots, g_m h_{m+n} g_m^{-1}) \\ &\quad + (-1)^m (g_1, \ldots, g_m) \times (\partial(h_{m+1}, \ldots, h_{m+n})) , \end{aligned}$$

where

$$\begin{aligned} \partial(g_1, \ldots, g_m) &= (g_2, \ldots, g_m) \\ + \sum_{i=1}^{m-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_m) &+ (-1)^m (g_1, \ldots, g_{m-1}) \\ &= \partial'(g_1, \ldots, g_m) + (-1)^m (g_1, \ldots, g_{m-1}) . \end{aligned}$$

For the above complex we triangulate it and associate the elements for their products.