Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

L'Enseignement Mathématique, t. 38 (1992), p. 329-344

RATIONALITY OF PIECEWISE LINEAR FOLIATIONS AND HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR HOMEOMORPHISMS

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INTRODUCTION

Let $\mathscr{F}$ be a codimension one transversely piecewise linear foliation of $S^3 \times S^3$. For such a foliation, the discrete Godbillon-Vey class is defined as a 3-dimensional cohomology class ([5], [3]). Hence in this case, $GV(\mathscr{F}) \in H^3(S^3 \times S^3; \mathbf{R}) \cong \mathbf{R} \oplus \mathbf{R}$.

In this paper, we first show that if $GV(\mathscr{F}) = (a, b) \in H^3(S^3 \times S^3; \mathbf{R})$, then $a/b \in \mathbf{Q} \cup \{\infty\}$, which is the meaning of the rationality in the title. The same question on the Godbillon-Vey class ([6]) for the smooth codimension one foliations was raised in Gel'fand-Feigin-Fuks [2] and discussed in Morita [10]. In the case of transversely oriented, transversely piecewise linear foliations, the classifying space for them is known by Greenberg ([7]). In fact, this classifying space is weakly homotopic to the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$ of two copies of $B\mathbf{R}^\delta = K(\mathbf{R}, 1)$ which is the classifying space for the additive group $\mathbf{R}$ with the discrete topology. Since the cup product is trivial on the cohomology ring of the join of two spaces (see § 1), the higher discontinuous invariants defined by Morita ([10]) are trivial in this classifying space. The rationality for codimension one transversely piecewise linear foliations of $S^3 \times S^3$ is a consequence of this.

Morita translated the question of rationality into that of graded commutativity of $*$-product defined on the homology of the group of diffeomorphisms of $\mathbf{R}$ with compact support ([10]). Using the description by Greenberg ([7]) of the classifying space for transversely oriented, transversely piecewise linear foliations, we can calculate the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support as well as the $*$-product structure. Then we see that the $*$-product is certainly not graded commutative, which insures the rationality. In fact we calculated this first, and later we found out the fact that the classifying space is a join is the origin of rationality.

This paper is organized as follows. In § 1, we show two lemmas in algebraic topology. One asserts that the cup product is trivial on the cohomology ring of the join of two spaces. The other concerns the relationship between the tensor product in the E^2 term of the spectral sequence associated to the fibration $\Omega X \to PX \to X$ and the Pontrjagin product on the homology of ΩX. Both of them should be well known but we include their proofs. In § 2, we review the definition of discontinuous invariants of Morita ([10]). We see immediately that all higher discontinuous invariants vanish for codimension one transversely piecewise linear foliations. This implies the rationality of such foliations. The rest of this paper concerns the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of the real line with compact support. This would be of interest because it would provide a good concrete example illustrating the relationship between the homology of the group of homeomorphisms and the homotopy of the classfying space for foliations. In § 3, we give the result of calculation of the homology of $PL_c(\mathbf{R})$. In § 4, we describe the way of calculation. This is done by defining sufficiently many cocycles. For this, we define and use a determinant with values in the tensor product over the rationals $\mathbf{Q}$ of a number of copies of $\mathbf{R}$. In § 5, we show the fact that the homomorphism $PL_c([0, \infty)) \to \mathbf{R}$ which sends f to $\log f'(0)$ induces a surjection in homology. Since there are no natural sections, this is not trivial. The nontriviality of the cocycles defined in § 4 depends on this fact.

My knowledge on the group of piecewise linear homeomorphisms of the real line was deepened during my visit à l'Université de Genève in the winter 1990/91. I would like to thank it for its warm hospitality. This work is done during my visit à l'Ecole Normale Supérieure de Lyon in the spring 1991. I would like to thank it for its warm hospitality and I also thank la Fondation Scientifique de Lyon et du Sud-Est for the financial support. I thank André Haefliger, Etienne Ghys, Peter Greenberg, Vlad Sergiescu and Shigeyuki Morita for their interest taken for this work.

§ 1. Lemmas

First we show the cup product is trivial on the cohomology ring of the join of two spaces. This is an exercise in algebraic topology.

LEMMA (1.1). *Let X and Y be two topological spaces. The cup product on the cohomology ring of the join $X * Y$ is trivial.*