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RATIONALITY OF PIECEWISE LINEAR FOLIATIONS AND HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR HOMEOMORPHISMS

by Takashi TSUBOI

INTRODUCTION

Let $\mathcal{F}$ be a codimension one transversely piecewise linear foliation of $S^3 \times S^3$. For such a foliation, the discrete Godbillon-Vey class is defined as a 3-dimensional cohomology class ([5], [3]). Hence in this case, $GV(\mathcal{F}) \in H^3(S^3 \times S^3; \mathbf{R}) \cong \mathbf{R} \oplus \mathbf{R}$.

In this paper, we first show that if $GV(\mathcal{F}) = (a, b) \in H^3(S^3 \times S^3; \mathbf{R})$, then $a/b \in \mathbf{Q} \cup \{\infty\}$, which is the meaning of the rationality in the title.

The same question on the Godbillon-Vey class ([6]) for the smooth codimension one foliations was raised in Gel'fand-Feigin-Fuks [2] and discussed in Morita [10]. In the case of transversely oriented, transversely piecewise linear foliations, the classifying space for them is known by Greenberg ([7]). In fact, this classifying space is weakly homotopic to the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$ of two copies of $B\mathbf{R}^\delta = K(\mathbf{R}, 1)$ which is the classifying space for the additive group $\mathbf{R}$ with the discrete topology. Since the cup product is trivial on the cohomology ring of the join of two spaces (see § 1), the higher discontinuous invariants defined by Morita ([10]) are trivial in this classifying space. The rationality for codimension one transversely piecewise linear foliations of $S^3 \times S^3$ is a consequence of this.

Morita translated the question of rationality into that of graded commutativity of $*$-product defined on the homology of the group of diffeomorphisms of $\mathbf{R}$ with compact support ([10]). Using the description by Greenberg ([7]) of the classifying space for transversely oriented, transversely piecewise linear foliations, we can calculate the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support as well as the $*$-product structure. Then we see that the $*$-product is certainly not graded commutative, which insures the rationality. In fact we calculated this first, and later we found out the fact that the classifying space is a join is the origin of rationality.

This paper is organized as follows. In § 1, we show two lemmas in algebraic topology. One asserts that the cup product is trivial on the cohomology ring of the join of two spaces. The other concerns the relationship between the tensor product in the E^2 term of the spectral sequence associated to the fibration $\Omega X \to PX \to X$ and the Pontrjagin product on the homology of ΩX. Both of them should be well known but we include their proofs. In § 2, we review the definition of discontinuous invariants of Morita ([10]). We see immediately that all higher discontinuous invariants vanish for codimension one transversely piecewise linear foliations. This implies the rationality of such foliations. The rest of this paper concerns the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of the real line with compact support. This would be of interest because it would provide a good concrete example illustrating the relationship between the homology of the group of homeomorphisms and the homotopy of the classfying space for foliations. In § 3, we give the result of calculation of the homology of $PL_c(\mathbf{R})$. In § 4, we describe the way of calculation. This is done by defining sufficiently many cocycles. For this, we define and use a determinant with values in the tensor product over the rationals $\mathbf{Q}$ of a number of copies of $\mathbf{R}$. In § 5, we show the fact that the homomorphism $PL_c([0, \infty)) \to \mathbf{R}$ which sends f to $\log f'(0)$ induces a surjection in homology. Since there are no natural sections, this is not trivial. The nontriviality of the cocycles defined in § 4 depends on this fact.

My knowledge on the group of piecewise linear homeomorphisms of the real line was deepened during my visit à l'Université de Genève in the winter 1990/91. I would like to thank it for its warm hospitality. This work is done during my visit à l'Ecole Normale Supérieure de Lyon in the spring 1991. I would like to thank it for its warm hospitality and I also thank la Fondation Scientifique de Lyon et du Sud-Est for the financial support. I thank André Haefliger, Etienne Ghys, Peter Greenberg, Vlad Sergiescu and Shigeyuki Morita for their interest taken for this work.

§ 1. Lemmas

First we show the cup product is trivial on the cohomology ring of the join of two spaces. This is an exercise in algebraic topology.

LEMMA (1.1). *Let X and Y be two topological spaces. The cup product on the cohomology ring of the join $X * Y$ is trivial.*

Proof. We may assume that X and Y are simplicial complexes. The simplices of the join $X * Y$ other than those in X and in Y are the joins of simplices of X and Y. Since $X * Y$ contains the cones of X and Y, any cocycle on $X * Y$ is cohomologous to a cocycle which vanishes on chains in X or Y. We look at the Alexander-Whitney approximation of the diagonal map $X * Y \to X * Y \times X * Y$. The image of a simplex in $X * Y$ in $C_*(X * Y) \otimes C_*(X * Y)$ is a sum of $\sigma_i \otimes \sigma_j$, where either σ_i or σ_j does not contain the edge corresponding to the joining interval. Hence the evaluation of the cup product of two modified cocycles is always zero.

The second lemma concerns the relationship between the tensor product in the E^2 term of the spectral sequence associated to the fibration $\Omega X \to PX \to X$ and the Pontrjagin product $*$ in the homology of the loop space ΩX.

LEMMA (1.2). *Let X be a simply connected CW complex such that $H_*(X; \mathbf{Z})$ is torsion free. Let PX and ΩX be the path space and the loop space of X, respectively. Let*

$$E^2_{p,q} = H_p(X; \mathbf{Z}) \otimes H_q(\Omega X; \mathbf{Z})$$

denote the E^2 term of the spectral sequence associated to the fibration. For positive integer p, there is a homomorphism

$$s: H_p(\Omega X; \mathbf{Z}) \to H_{p+1}(X; \mathbf{Z})$$

such that, for $v \in H_q(\Omega X; \mathbf{Z})$,

$$s(u) \otimes v \in E^2_{p+1,q} = H_{p+1}(X; \mathbf{Z}) \otimes H_q(\Omega X; \mathbf{Z})$$

and

$$u * v \in H_{p+q}(\Omega X; \mathbf{Z})$$

are related under ∂^{p+1}, where $*$ denotes the (Pontrjagin) product induced by the composition of loops. More precisely, for the submodules $Z^r_{p+1,q}$ and $B^r_{p+1,q}$ of $E^2_{p+1,q}$ which give $E^r_{p+1,q} = Z^r_{p+1,q}/B^r_{p+1,q}$,

$$s(u) \otimes v \in Z^p_{p+1,q} \quad \text{and} \quad \partial^{p+1}(s(u) \otimes v) - u * v \in B^p_{0,p+q}.$$

Proof. The element u is represented by the image of the fundamental class of a p-dimensional finite complex Y under a continuous map $Y \to \Omega X$. We define $s(u)$ to be the class represented by the adjoint map $SY \to X$, where SY denotes the suspension of Y. Since the composition $Y \to \Omega X \to PX$ bounds the map $SY \to PX$ in the obvious way and the composition $SY \to PX \to X$ represents $s(u) \in H_{p+1}(X; \mathbf{Z}), s(u)$ and u are related under ∂^{p+1}. Let $Z \to \Omega X$ represent v. Consider the composition

$$Y \times Z \to \Omega X \times \Omega X \overset{*}{\to} \Omega X .$$

Then this represents $u * v \in H_{p+q}(\Omega X; \mathbf{Z})$. On the other hand, the composition

$$Y \times Z \to \Omega X \times \Omega X \to PX \times \Omega X \to X \times \Omega X$$

bounds $SY \times Z \to PX \times \Omega X \to X \times \Omega X$, which represents $s(u) \otimes v$. Hence $s(u) \otimes v$ and $u * v$ are related under ∂^{p+1}.

§2. Discontinuous invariants

First we review the definition by Morita ([10]) of discontinuous invariants arising from the Godbillon-Vey invariant for codimension one foliations.

Let $\mathscr{F}$ be a codimension one foliation of a closed oriented $3k$-dimensional manifold M. Then the Godbillon-Vey class $gv(\mathscr{F}) \in H^3(M; \mathbf{R})$ is defined ([6]). Let $\{x_1, \ldots, x_n\}$ be a basis of $H^3(M; \mathbf{Q})$. Then $gv(\mathscr{F})$ is written as

$$gv(\mathscr{F}) = a_1 x_1 + \ldots + a_n x_n ,$$

where $a_1, \ldots, a_n \in \mathbf{R}$. The discontinuous invariant GV_k is defined by

$$GV_k(\mathscr{F}) = \sum_{i_1 < \ldots < i_k} (x_{i_1} \cup \ldots \cup x_{i_k})[M]\ a_{i_1} \wedge_{\mathbf{Q}} \ldots \wedge_{\mathbf{Q}} a_{i_k} \in \mathbf{R}^{\wedge k} = \overbrace{\mathbf{R} \wedge_{\mathbf{Q}} \ldots \wedge_{\mathbf{Q}} \mathbf{R}}^{k} ,$$

where $[M] \in H_{3k}(M; \mathbf{Z})$ is the fundamental class. Morita showed that GV_k is natural, GV_k depends only on the foliated cobordism class of $\mathscr{F}$, and hence there is a universal map $GV_k \colon H_{3k}(B\Gamma_1; \mathbf{Z}) \to \mathbf{R}^{\wedge k}$ ([10]).

The same argument applies to transversely piecewise linear foliations and the discrete Godbillon-Vey class defined in [5] and [3]. Then the following theorem is obtained from the description by Greenberg ([7]) of the classifying space for them and Lemma (1.1).

THEOREM (2.1). *Let $\mathscr{F}$ be a codimension one transversely orientable transversely piecewise linear foliation of a closed oriented $3k$-dimensional manifold $M (k \geqslant 2)$. Then $GV_k(\mathscr{F}) = 0$.*

Proof. The weak homotopy type of the classifying space $B\bar{\Gamma}_1^{PL}$ for codimension one transversely oriented transversely piecewise linear foliations is known by Greenberg ([7]). This classifying space $B\bar{\Gamma}_1^{PL}$ has the weak homotopy type of the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$ of two copies of $B\mathbf{R}^\delta = K(\mathbf{R}, 1)$. Let

gv denote the discrete Godbillon-Vey class defined as a 3-dimensional cohomology class of this classifying space ([5], [3]).

$$gv \in H^3(B\bar{\Gamma}_1^{PL}; \mathbf{R}) .$$

By Lemma (1.1), the higher discontinuous invariants GV_k are trivial in this classifying space $B\bar{\Gamma}_1^{PL}$. Hence by the naturality of GV_k, $GV_k(\mathscr{F}) = 0$.

COROLLARY (2.2). *Let $\mathscr{F}$ be a codimension one transversely piecewise linear foliation of $S^3 \times S^3$. $GV(\mathscr{F}) = (a, b) \in H^3(S^3 \times S^3, \mathbf{R})$ satisfies $a/b \in \mathbf{Q} \cup \{\infty\}$.*

Proof. $0 = GV_2(\mathscr{F}) = a \wedge_{\mathbf{Q}} b$. Hence $a/b \in \mathbf{Q} \cup \{\infty\}$.

Remark. Morita translated the question of rationality into that of graded commutativity of $*$-product defined on the homology of the group of diffeomorphisms of $\mathbf{R}$ with compact support ([10]). In the later sections, we calculate the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support as well as the $*$-product structure. We see that the $*$-product is certainly not graded commutative, which insures the rationality. The argument on the rationality of transversely piecewise linear foliations uses the fact that the Godbillon-Vey invariant localizes on transversely discrete sets and this argument cannot be generalized for smooth foliations for the moment. See how the class $C^4_{(1,1,1,1)}$ exists in §3. We also see that the Whitehead product of elements of $\pi_n(B\bar{\Gamma}_1^{PL})$ which are not zero in homology is usually nontrivial and has infinite order.

Remark. The Hurewicz map

$$\pi_n(B\bar{\Gamma}_1^{PL}) \to H_n(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$$

is surjective. To see this, note first that by Greenberg ([7]),

$$H_n(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \cong \sum_{i=1}^{n-1} \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge n-1-i} .$$

An element $(a_1 \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} a_i) \otimes_{\mathbf{Q}} (b_{i+1} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} b_{n-1}) \in \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge n-1-i}$ is represented by the following foliation of $T^i * T^{n-1-i}$. Consider the foliated $\mathbf{R}$-product with noncompact support over T^{n-1} such that the holonomy $h: \pi_1(T^{n-1}) \to PL(\mathbf{R})$ is given by

$$h(e_j)(x) = e^{a_i}x \text{ for } x < 0 \text{ and } h(e_j)(x) = x \text{ for } x > 0 \text{ if } j = 1, \dots, i$$

$$h(e_j)(x) = x \text{ for } x < 0 \text{ and } h(e_j)(x) = e^{b_j}x \text{ for } x > 0 \text{ if } j = i+1, \dots, n-1 .$$

This foliation restricted to $T^{n-1} \times [-1, 1]$ induces a foliation of $T^i * T^{n-1-i}$ which is

$$T^{n-1} \times [-1, 1]/(T^i \times T^{n-1-i} \times \{-1\} \sim T^i \times \{-1\}\,,$$
$$T^i \times T^{n-1-i} \times \{1\} \sim T^{n-1-i} \times \{1\})\,.$$

Note that there is a degree one map from the suspension of T^{n-1} to $T^i * T^{n-1-i}$. Since we can embed $T^{n-1} \times [-1, 1]$ in S^n, we have a degree one map from S^n to the suspension of T^{n-1}, hence to $T^i * T^{n-1-i}$. Thus Hurewicz map is surjective.

§3. Homology of the group of piecewise linear homeomorphisms

Let $PL_c(\mathbf{R})$ denote the group of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support. Let $\mu: PL_c(\mathbf{R}) \times PL_c(\mathbf{R}) \to PL_c(\mathbf{R})$ be the composition of two isomorphisms $PL_c(\mathbf{R}) \cong PL_c((-\infty, 0))$ and $PL_c(\mathbf{R}) \cong PL_c((0, \infty))$, and the inclusion

$$PL_c((-\infty, 0)) \times PL_c((0, \infty)) \to PL_c(\mathbf{R})\,.$$

Then μ induces a product $*$ on the homology of $BPL_c(\mathbf{R})^\delta$ ([10]).

The homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support is described as follows. For positive integers i and j, put

$$V^{i,j} = \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge j}$$
$$= \overbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}^{i} \otimes_{\mathbf{Q}} \overbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}^{j}\,.$$

THEOREM (3.1).

$$H_m(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong \sum V^{k_1^-, k_1^+} \otimes_{\mathbf{Q}} \dots \otimes_{\mathbf{Q}} V^{k_s^-, k_s^+}\,,$$

where the sum is taken over even number of positive integers

$$(k_1^-, k_1^+, \dots, k_s^-, k_s^+)$$

such that $k_1^- + k_1^+ + \dots + k_s^- + k_s^+ = m$. *Moreover, the $*$-product*

$$*: H_i(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \times H_j(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \to H_{i+j}(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$$

coincides with the tensor product.

For small dimensions, this theorem says that

$$H_1(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong 0\,,$$
$$H_2(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}\,,$$
$$H_3(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{Q}} \mathbf{R} \oplus \mathbf{R} \otimes_{\mathbf{Q}} (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}), \text{ and}$$
$$H_4(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{Q}} \mathbf{R} \oplus (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{Q}} (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \oplus \mathbf{R} \otimes_{\mathbf{Q}} (\mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R} \wedge_{\mathbf{Q}} \mathbf{R}) \oplus \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}\,.$$

The first homology group is 0 is equivalent to that $PL_c(\mathbf{R})$ is perfect and this is due to Epstein ([1]). The second and third homologies are given explicitly by Greenberg ([7]). The summand $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ of $H_4(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ is the image of the $*$-product on $H_2(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ and since the $*$-product coincides with the tensor product, the $*$-product is not graded commutative. This implies that the Whitehead product

$$\pi_3(B\bar{\Gamma}_1^{PL}) \times \pi_3(B\bar{\Gamma}_1^{PL}) \to \pi_5(B\bar{\Gamma}_1^{PL})$$

is highly nontrivial and this is the obstruction to construct a foliation on $S^3 \times S^3$ with given Godbillon-Vey class. In this way, as in mentioned in §2, this is related to the rationality (see [10]).

Theorem (3.1) is also obtained as an application of the description by Greenberg ([7]) of the classifying space $B\bar{\Gamma}_1^{PL}$. As we mentioned, his result says that this classifying space is weakly homotopy equivalent to the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$. To show Theorem (3.1), we use the isomorphism

$$H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong H_*(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z})$$

due to Mather ([9]) adapted for the PL case by Ghys-Sergiescu ([5]) and Greenberg ([7]) using a result of Segal ([11]), and the homology spectral sequence associated to the fibration

$$\Omega B\bar{\Gamma}_1^{PL} \to PB\bar{\Gamma}_1^{PL} \to B\bar{\Gamma}_1^{PL}\,.$$

Since $B\bar{\Gamma}_1^{PL}$ is simply connected, the E^2 term of this spectral sequence is as follows.

$$E^2_{p+1,q} = H_{p+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z})\,.$$

Note that $H_*(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$ is torsion free. From this, Greenberg obtained the second and the third homologies ([7]). To show our theorem, we show that, for $p \geqslant 0$,

$$E^2_{p+1,q} = Z^2_{p+1,q} = \dots = Z^p_{p+1,q} \quad \text{and}$$
$$Z^{p+1}_{p+1,q} = Z^\infty_{p+1,q} = 0 = B^\infty_{p+1,q} = \dots = B^2_{p+1,q}\,.$$

This is equivalent to that the differentials induce an isomorphism

$$\sum_{p+q=m,p\geqslant 0} H_{p+1}(B\bar{\Gamma}_1^{PL};\mathbf{Z})\otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z}) \to H_{p+q}(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z})\,.$$

To show this we define the cohomology classes of $BPL_c(\mathbf{R})^\delta$ which detect the images of generators of $H_{p+1}(B\bar{\Gamma}_1^{PL};\mathbf{Z})\otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z})$.

§4. Construction of cocycles of the group $PL_c(\mathbf{R})$

Tensor determinants. We define a determinant of an $(n\times n)$ real matrix which takes values in the tensor product over $\mathbf{Q}$ of n copies of $\mathbf{R}$. For $(a_{ij})_{i,j=1,\dots,n}$, we put

$$\det{}^{\otimes_{\mathbf{Q}}}(a_{ij}) = \sum_\sigma \operatorname{sign}(\sigma)\, a_{\sigma(1)1}\otimes_{\mathbf{Q}}\cdots\otimes_{\mathbf{Q}} a_{\sigma(n)n}\,.$$

For example,

$$\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = a_{11}\otimes_{\mathbf{Q}} a_{22} - a_{21}\otimes_{\mathbf{Q}} a_{12}\,.$$

We have the usual multilinearity but we do not have the usual alternativity. For example,

$$\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix} a & b\\ a & b\end{pmatrix} = 0 \quad\text{but}\quad \det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix} a & a\\ b & b\end{pmatrix} = a\otimes_{\mathbf{Q}} b - b\otimes_{\mathbf{Q}} a = a\wedge_{\mathbf{Q}} b\,.$$

The latter is not necessarily zero. In general, if we change the rows then this determinant changes sign, however, there are no simple laws for changing columns. It is worth noticing that we have the usual formula of developing with respect to the first or the last column.

$$\begin{aligned}\det{}^{\otimes_{\mathbf{Q}}}(a_{ij}) &= \sum_{i+1}^{n} (-1)^{i+1} a_{i1}\otimes_{\mathbf{Q}} \det{}^{\otimes_{\mathbf{Q}}}(A_{i1})\\ &= \sum_{i+1}^{n} (-1)^{i+n} \det{}^{\otimes_{\mathbf{Q}}}(A_{in})\otimes_{\mathbf{Q}} a_{in}\,,\end{aligned}$$

where A_{ij} is the matrix (a_{ij}) with the i-th row and the j-th column deleted.

Cocycles of Lipschitz homeomorphism groups. We review the construction of cocycles of certain Lipschitz homeomorphism groups of the real line or the circle (see [13]). Let $\mathscr{S}$ be the space of functions with compact support which are locally constant outside of finitely many points. (For other Lipschitz homeomorphism groups, $\mathscr{S}$ is replaced by other spaces of functions which contains the logarithm of derivatives of the homeomorphisms.) Let V be a **Q**-vector space. Let

$$A: \overbrace{\mathscr{S} \times \ldots \times \mathscr{S}}^{n} \to V$$

be a multilinear form which is invariant under the parameter change in the following sense. If h is a homeomorphism of **R** with compact support, then

$$A(\varphi_1 \circ h, \ldots, \varphi_n \circ h) = A(\varphi_1, \ldots, \varphi_n) .$$

Then the V valued function

$$C: \overbrace{PL_c(\mathbf{R}) \times \ldots \times PL_c(\mathbf{R})}^{n} \to V$$

defined by

$$C(g_1, g_2, \ldots, g_n) = A(\log g_1' \circ g_2 \circ \ldots \circ g_n, \log g_2' \circ g_3 \circ \ldots \circ g_n, \ldots, \log g_n')$$

is an n-cocycle of $PL_c(\mathbf{R})$. The verification is straightforward.

Cocycles of PL homeomorphism groups. For a $(2s)$-tuple of positive integers $(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)$ such that $k_1^- + k_1^+ + \ldots + k_s^- + k_s^+ = m$, we define a multilinear form

$$A^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}: \overbrace{\mathscr{S} \times \ldots \times \mathscr{S}}^{m} \to \mathbf{R}^{\otimes m}$$

whose values are contained in $V^{k_1^-, k_1^+} \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} V^{k_s^-, k_s^+}$. This is given by

$$A^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}(\varphi_1, \ldots, \varphi_m) = \sum_{x_1 < \ldots < x_s} \frac{1}{m!} \det{}^{\otimes_{\mathbf{Q}}}$$

$$(\underbrace{\varphi(x_1 - 0) \ldots \varphi(x_1 - 0)}_{k_1^-} \; \underbrace{\Delta\varphi(x_1) \ldots \Delta\varphi(x_1)}_{k_1^+}$$

$$\ldots \underbrace{\varphi(x_s - 0) \ldots \varphi(x_s - 0)}_{k_s^-} \; \underbrace{\Delta\varphi(x_s) \ldots \Delta\varphi(x_s)}_{k_s^+}) ,$$

where φ denotes the vertical vector ${}^t(\varphi_1, \ldots, \varphi_m)$ and $\Delta\varphi(x) = \varphi(x+0) - \varphi(x-0)$. Note that, since $\varphi_1, \ldots, \varphi_m$ are elements of $\mathscr{S}$, the sum is in fact a finite sum. It is clear that $A^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$ is invariant under the parameter change.

For example, the functional $A^2_{(1,1)}: \mathscr{S} \times \mathscr{S} \to \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$ is defined as follows.

$$A^2_{(1,1)}(\varphi_1, \varphi_2) = \sum_{x \in \mathbf{R}} \frac{1}{2} \det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} \varphi_1(x-0) & \Delta\varphi_1(x) \\ \varphi_2(x-0) & \Delta\varphi_2(x) \end{pmatrix}$$
$$= \sum_{x \in \mathbf{R}} \frac{1}{2} (\varphi_1(x-0) \otimes_{\mathbf{Q}} \Delta\varphi_1(x) - \varphi_2(x-0) \otimes_{\mathbf{Q}} \Delta\varphi_2(x)) \in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} .$$

Then $A^2_{(1,1)}$ is bilinear and invariant under the parameter change. This functional $A^2_{(1,1)}$ composed with the evaluation map $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \to \mathbf{R}$ gives the area of the polygon whose vertices are the image of (φ_1, φ_2) and whose edges join the subsequent vertices with respect to the order of $\mathbf{R}$. The functional $A^2_{(1,1)}$ gives rise to the following 2-cocycle $C^2_{(1,1)}$.

$$C^2_{(1,1)}(g_1, g_2) = \sum_{x \in \mathbf{R}} \frac{1}{2} \det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} \log g_1' \circ g_2(x-0) & \Delta \log g_1' \circ g_2(x) \\ \log g_2'(x-0) & \Delta \log g_2'(x) \end{pmatrix} \in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} .$$

This 2-cocycle $C^2_{(1,1)}$ composed with the evaluation map $\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \to \mathbf{R}$ is the discrete Godbillon-Vey invariant ([5], [3], [13], [8]).

The nontriviality of this class is shown easily. Let g_1 and g_2 be piecewise linear homeomorphisms of $\mathbf{R}$ with support in $[-1, 0]$ and $[0, 1]$, respectively, such that $\log g_1'(0-0) = a$ and $\log g_2'(0+0) = b$. Then $(g_1, g_2) - (g_2, g_1)$ is a 2-cycle and

$$C^2_{(1,1)}((g_1, g_2) - (g_2, g_1)) = \frac{1}{2} \det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} a & -a \\ 0 & b \end{pmatrix} - \frac{1}{2} \det{}^{\otimes_{\mathbf{Q}}} \begin{pmatrix} 0 & b \\ a & -a \end{pmatrix}$$
$$= a \otimes_{\mathbf{Q}} b .$$

Another interesting example is $A^4_{(1,1,1,1)}$ defined by

$$A^4_{(1,1,1,1)}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \sum_{x<y} \frac{1}{4!} \det{}^{\otimes_{\mathbf{Q}}}(\varphi(x-0)\Delta\varphi(x)\varphi(y-0)\Delta\varphi(y))$$
$$\in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$$

where φ denotes the vertical vector ${}^t(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$. This gives rise to the cocycle $C^4_{(1,1,1,1)}$ which measures the noncommutativity of $*$-product. The nontriviality of $C^4_{(1,1,1,1)}$ is easily shown by evaluating on the $*$-product of two examples described above.

Independence of the cohomology classes. The bijectivity of the homomorphism

$$\sum_{p+q=m} H_{p+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \to H_{p+q}(\Omega B\bar{\Gamma}_1^{PL}; \mathbf{Z})$$

is equivalent to the independence of the cohomology classes of these cocycles $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$. To show the independence we use the following theorem.

THEOREM (4.1). *Let* $j_+ : PL_c([0, \infty)) \to \mathbf{R}$ *denote the homomorphism defined by*

$$j_+(f) = \log f'(0) .$$

The homomorphism j_+ *induces a surjection in integer homology.*

Using this theorem, we can show the independence. Let $u_i^- \otimes_{\mathbf{Q}} u_i^+$ be an element of $V^{k_i^-, k_i^+}$ ($u_i^- \in \mathbf{R}^{\wedge k_i^-}$, $u_i^+ \in \mathbf{R}^{\wedge k_i^+}$). Then we have a k_i^--dimensional cycle σ_i^- of $BPL_c((-\infty, 0])^\delta$ such that the image under $(j_-)_*$ coincides with $u_i^- \in \mathbf{R}^{\wedge k_i^-} \cong H_{k_i^-}(B\mathbf{R}^\delta; \mathbf{Z})$, where $j_- : PL_c((-\infty, 0]) \to \mathbf{R}$ denotes the homomorphism defined by $j_-(f) = \log f'(0)$. We also have a k_i^+-dimensional cycle σ_i^+ of $BPL_c([0, \infty))^\delta$ such that the image under $(j_+)_*$ coincides with $u_i^+ \in \mathbf{R}^{\wedge k_i^+} \cong H_{k_i^+}(B\mathbf{R}^\delta; \mathbf{Z})$. Then $\sigma_i^- \times \sigma_i^+$ is a $(k_i^- + k_i^+)$-dimensional cycle of $B(PL_c((-\infty, 0]) \times PL_c([0, \infty)))^\delta$ such that the image under $(j_- \times j_+)_*$ coincides with $u_i^- \otimes_{\mathbf{Q}} u_i^+ \in V^{k_i^-, k_i^+}$. Now let $T_1, \ldots, T_s$ be translations of $\mathbf{R}$ such that $T_1(0) < \ldots < T_s(0)$ and the supports of $\sigma_i = T_i(\sigma_i^- \times \sigma_i^+) T_i^{-1}$ are contained in disjoint open intervals, where the support of a cycle of $BPL_c(\mathbf{R})^\delta$ is the union of the supports of the homeomorphisms which appear in the expression of the cycle. Then $\sigma_1 \times \ldots \times \sigma_s$ is an m-cycle and the value of the cocycle $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$ on it is $(u_1^- \otimes_{\mathbf{Q}} u_1^+) \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} (u_s^- \otimes_{\mathbf{Q}} u_s^+)$. It is easy to see that the values of the other m-cocycles on this cycle are 0.

The fact that $*$-product coincides with the tensor product follows from Lemma (1.2). Note that the map s in Lemma (1.2) is an isomorphism from the subgroup of $H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ generated by the $\sigma^- \times \sigma^+$ to $H_{*+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$. Thus Theorem (3.1) is proved.

§5. SURJECTIVITY OF $(j_+)_*$

We prove Theorem (4.1). We consider j_+ as a homomorphism from $PL_c([0, \infty))$ to the group of germs at 0. We use the fact that the n-dimensional homology group of $B\mathbf{R}^\delta$ is isomorphic to $\mathbf{R}^{\wedge n}$ and whose generators are represented by the images of the fundamental classes of tori T^n of dimension n under the mappings which are defined by n (commuting) elements. We will construct an n-complex Y_n with the fundamental class and a degree one map $Y_n \to T^n$. Then for each mapping $T^n \to B\mathbf{R}^\delta$, we will construct a mapping $Y_n \to BPL_c([0,\infty))^\delta$ such that the following diagram commutes.

$$\begin{array}{ccc} Y_n & \to & BPL_c([0,\infty))^\delta \\ \downarrow & & \downarrow \\ T^n & \to & B\mathbf{R}^\delta . \end{array}$$

Theorem (4.1) follows immediately from this commutative diagram.

Construction of Y_n. Let L be a large positive real number. In the Euclidean n space, we consider the following polyhedron X_n

$$X_n = \{(x_1, \dots, x_n) \in [0, L]^n ; \quad x_{i_1} + \dots + x_{i_k} \geqslant (k-1)k/2 \quad \text{for} \quad 1 \leqslant i_1 < \dots < i_k \leqslant n\} .$$

The shape of X_n is the cube with certain neighborhoods of the k-faces ($k \leqslant n-2$) in the coordinate planes deleted, those of the $(k-1)$-faces being thicker than those of the k-faces.

The polyhedron X_n has $2^n - 1 + n$ faces of dimension $n-1$. If $(x_1, \dots, x_n)$ is a vertex of X_n then $(x_1, \dots, x_n)$ is a permutation of $(0, 1, .., k, L, \dots, L)$. In this case we say $(x_1, \dots, x_n)$ is a vertex of type $\{0, 1, .., k, L, \dots, L\}$. There are edges between $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ of the same type $\{0, 1, .., k, L, \dots, L\}$ if one is obtained from the other by permuting two coordinates. The edges between different types exists only if the types are $\{0, 1, .., k-1, L, \dots, L\}$ and $\{0, 1, .., k, L, \dots, L\}$, and one vertex is obtained from the other by changing the entries k and L.

The polyhedron X_n has the $(n-1)$-face $\{x_i = L\}$ which is isometric to X_{n-1}. The $(n-1)$-face $\{x_i = 0\}$ is isometric to X_{n-1} with L replaces by $L-1$ because if $x_i = 0$ then

$$x_{i_1} + \dots + x_{i_k} \geqslant (k-1)k/2$$

for $\{i_1, \dots, i_k\}$ containing i implies

$$(x_{i_1} - 1) + \dots + (x_{i_k} - 1) \geqslant (k-1)k/2$$

for $\{i_1, \dots, i_k\}$ not containing i. Hence we can define a simplicial identification between the faces $\{x_i = L\}$ and $\{x_i = 0\}$. In general, the face

$$\{x_{i_1} + \dots + x_{i_k} = (k-1)k/2\}$$

is isometric to $X'_{n-k} \times \Sigma_k$, where X'_{n-k} is X_{n-k} with L replaced by $L-k$ and Σ_k is the face $\{x_1 + \ldots + x_k = (k-1)k/2\}$ in X_k. The reason is

$$x_{i'_1} + \ldots + x_{i'_{k'}} \geqslant (k'-1)k'/2$$

for $\{i'_1, \ldots, i'_{k'}\}$ containing $\{i_1, \ldots, i_k\}$ implies

$$(x_{i'_1} - k) + \ldots + (x_{i'_{k'}} - k) \geqslant (k'-1)k'/2$$

for $\{i'_1, \ldots, i'_{k'}\}$ not containing $\{i_1, \ldots, i_k\}$. We also fix a simplicial identification between X'_{n-k} and X_{n-k}. Now we distinguish the faces by the set $\{i_1, \ldots, i_k\}$ of indices and we see that

$$\partial X_n = \bigcup_{A \subset \{1, \ldots, n\},\ \# A \geqslant 2} X_{\{1, \ldots, n\} - A} \times \Sigma_A \cup \bigcup_i X^{(L)}_{\{1, \ldots, n\} - \{i\}} \cup \bigcup_i X^{(0)}_{\{1, \ldots, n\} - \{i\}} ,$$

where

$$X_{\{1, \ldots, n\} - A} \times \Sigma_A = \{x_{i_1} + \ldots + x_{i_k} = (k-1)k/2\} \quad \text{if} \quad A = \{i_1, \ldots, i_k\} ,$$
$$X^{(L)}_{\{1, \ldots, n\} - \{i\}} = \{x_i = L\} \quad \text{and} \quad X^{(0)}_{\{1, \ldots, n\} - \{i\}} = \{x_i = 0\} .$$

The complex Y_n is defined inductively as follows. $Y_1 = X_1 = [0, L]$. Y_2 is obtained from X_2 (a pentagon) by identifying $X^{(L)}_{\{i\}}$ and $X^{(0)}_{\{i\}} (i = 1, 2)$ and by taking the double of it. Hence Y_2 is a surface of genus 2. We call the new part in the double $B\Sigma_{\{1,2\}}$.

$$Y_2 = X_2 + B\Sigma_{\{1,2\}} .$$

Y_3 is obtained from X_3 by identifying $X^{(L)}_{\{i,j\}}$ and $X^{(0)}_{\{i,j\}} (i, j = 1, 2, 3)$, by attaching $X_{\{k\}} \times B\Sigma_{\{i,j\}} (\{i, j, k\} = \{1, 2, 3\})$ to each $X_{\{k\}} \times \Sigma_{\{i,j\}}$, and then by taking the double. The boundary before taking the double is a surface of genus 6. We call the new part in the double $B\Sigma_{\{1,2,3\}}$.

$$Y_3 = X_3 + \sum_{\{i_1, i_2\} \subset \{1,2,3\}} X_{\{1,2,3\} - \{i_1, i_2\}} \times B\Sigma_{\{i_1, i_2\}} + B\Sigma_{\{1,2,3\}} .$$

In general, we define Y_n to be the double of

$$X_n + \sum_{A \subset \{1, \ldots, n\},\ \# A \geqslant 2} X_{\{1, \ldots, n\} - A} \times B\Sigma_A$$

and we call the new part in the double $B\Sigma_{\{1, \ldots, n\}}$.

$$Y_n = X_n + \sum_{A \subset \{1, \ldots, n\},\ \# A \geqslant 2} X_{\{1, \ldots, n\} - A} \times B\Sigma_A + B\Sigma_{\{1, \ldots, n\}} .$$

The mapping from Y_n to T^n is the one which sends the all $B\Sigma_A$ parts to a point and X_n to the fundamental domain of T^n.

Construction of $Y_n \to BPL_c([0,\infty))^\delta$. Now given a mapping $T^n \to B\mathbf{R}^\delta$, we construct a mapping $Y_n \to BPL_c([0,\infty))^\delta$. In other words, given a homomorphism $\mathbf{Z}^n \to \mathbf{R}$, we construct a homomorphism $\pi_1(Y_n) \to PL_c([0,\infty))$. This is also done inductively.

For $n = 1$, it is only necessary to choose a lift in $PL_c([0,\infty))$ of an element of $\mathbf{R}$.

Now for $n = 2$, we choose lifts f_1, f_2 of the generators of $\mathbf{Z}^2$. To the edges of Y_2, we associate elements of $PL_c([0,\infty))$. We put f_1 on the edges of X_2 from (L,L) to $(0,L)$ and from $(L,0)$ to $(1,0)$, and we put f_2 on the edges of X_2 from (L,L) to $(L,0)$ and from $(0,L)$ to $(0,1)$. Then we put the commutator $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$ on the edge from $(0,1)$ to $(1,0)$. Note that the support of this commutator does not contain 0 hence this commutator is an element of $PL_c((0,\infty))$. This commutator is also written as a commutator of elements of $PL_c((0,\infty))$. We can do it very easily, not by using the perfectness of the group $PL_c((0,\infty))$, but by using a conjugation by an element of $PL_c(\mathbf{R})$ which sends 0 to $a(>0)$ and which is the identity on $(2a,\infty)$ when the support of $[f_1, f_2]$ is contained in $(2a,\infty)$. We call this conjugation c_*. (This technique using conjugation is similar to that in [12].) c_* is an isomorphism from $PL_c([0,\infty))$ to a subgroup of $PL_c((0,\infty))$. Then $[f_1, f_2] = c_*([f_1, f_2]) = [c_* f_1, c_* f_2]$ and we associate $c_* f_1, c_* f_2$ to the edges in the new part in the double (in the mirror). Thus we defined the desired mapping $Y_2 \to BPL_c([0,\infty))^\delta$.

For general n, we use the same strategy. First we choose lifts $f_1, \ldots, f_n$ of the generators of $\mathbf{Z}^n$. To the edges of X_n, we associate elements of $PL_c([0,\infty))$. We associate f_i to the edge from a vertex of type $\{0,1,..,k-1,L,\ldots,L\}$ to a vertex of type $\{0,1,..,k,L,\ldots,L\}$ if the i-th coordinate changes from L to k. Then the elements associated to other edges are uniquely determined. In fact, we can associate an element of $PL_c([0,\infty))$ to each vertices as follows. We associate id to the vertex of type $\{L,\ldots,L\}$, if we already associated an element f_v to a vertex v of type $\{0,1,..,k-1,L,\ldots,L\}$ and a vertex v' is obtained from v by changing the i-th coordinate from L to k then we associate $f_i f_v$ to the vertex v'. Thus the edge from one vertex v_1 to another vertex v_2 is associated with $f_{v_2} f_{v_1}^{-1}$. Now if we look at the edges of Σ_A in the $(n-1)$-face $X_{\{1,\ldots,n\}-A} \times \Sigma_A$ the associated elements are in $PL_c((0,\infty))$. By induction, we can find $B\Sigma_A$ with edges in $PL_c((0,\infty))$. Thus we find the boundary of

$$X_n + \sum_{A \subset \{1,\ldots,n\},\ \# A \geqslant 2} X_{\{1,\ldots,n\}-A} \times B\Sigma_A$$

is a cycle of $PL_c((0, \infty))$. Here the products are considered as in the following remark. Hence in the double Y_n, we can associate the images under c_* in the new part of the double. (c_* is the conjugation by an element of $PL_c(\mathbf{R})$ which sends 0 to $a'(>0)$ and which is the identity on $(2a', \infty)$ when the support of the above boundary is contained in $(2a', \infty)$.) Thus we defined the desired mapping $Y_n \to BPL_c([0, \infty))^\delta$. This proves Theorem (4.1).

Remark. For two simplices $(g_1, \ldots, g_m)$ and $(h_{m+1}, \ldots, h_{m+n})$ of the classifying space for a discrete group, we define the product of them as follows.

$$(g_1, \ldots, g_m) \times (h_{m+1}, \ldots, h_{m+n}) = \sum_\sigma \operatorname{sign}(\sigma)\, (f_{\sigma,1}, \ldots, f_{\sigma,m+n}) \,.$$

where the sum is taken over the shuffles σ (that is, those permutations such that $\sigma(1) < \ldots < \sigma(m)$ and $\sigma(m+1) < \ldots < \sigma(m+n)$. The entry $f_{\sigma,j}$ is defined as follows.

$$f_{\sigma,\sigma(j)} = g_j \quad (j = 1, \ldots, m) \quad \text{and}$$
$$f_{\sigma,m+j} = (g_k \cdots g_m) h_{m+j} (g_k \cdots g_m)^{-1} \quad (j = 1, \ldots, n) \,,$$

where k is the integer such that $\sigma(k-1) < \sigma(m+j) < \sigma(k)$. For example, $(g_1, g_2) \times (h_3, h_4)$

$$\begin{aligned} &= (g_1, g_2, h_3, h_4) - (g_1, g_2 h_3 g_2^{-1}, g_2, h_4) \\ &\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2, h_4) + (g_1, g_2 h_3 g_2^{-1}, g_2 h_4 g_2^{-1}, g_2) \\ &\quad - (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2 h_4 g_2^{-1}, g_2) \\ &\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1 g_2 h_4 (g_1 g_2)^{-1}, g_1, g_2). \end{aligned}$$

This product is defined so that

$$\begin{aligned} &\partial((g_1, \ldots, g_m) \times (h_{m+1}, \ldots, h_{m+n})) \\ &= (\partial'(g_1, \ldots, g_m)) \times (h_{m+1}, \ldots, h_{m+n}) \\ &\quad + (-1)^m (g_1, \ldots, g_{m-1}) \times (g_m h_{m+1} g_m^{-1}, \ldots, g_m h_{m+n} g_m^{-1}) \\ &\quad + (-1)^m (g_1, \ldots, g_m) \times (\partial(h_{m+1}, \ldots, h_{m+n})) \,, \end{aligned}$$

where

$$\begin{aligned} \partial(g_1, \ldots, g_m) &= (g_2, \ldots, g_m) \\ + \sum_{i=1}^{m-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_m) &+ (-1)^m (g_1, \ldots, g_{m-1}) \\ &= \partial'(g_1, \ldots, g_m) + (-1)^m (g_1, \ldots, g_{m-1}) \,. \end{aligned}$$

For the above complex we triangulate it and associate the elements for their products.

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