

Objektyp: **Group**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*  
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, [www.library.ethz.ch](http://www.library.ethz.ch)

<http://www.e-periodica.ch>

The error term in (2) is best possible, and for the range  $1/2 < \sigma < 3/4$  the above asymptotic formula has been considerably refined by K. Matsumoto [5] (see also [4]).

It seems that the case  $\sigma = 1$  in (1) has not received much attention in the literature. The aim of this paper is to discuss this problem. We shall prove the following

**THEOREM.** *Let for  $T > 1$  the function  $R(T)$  be defined by*

$$(3) \quad \int_1^T |\zeta(1+it)|^2 dt = \zeta(2)T - \pi \log T + R(T).$$

Then

$$(4) \quad R(T) = O(\log^{2/3} T (\log \log T)^{1/3}),$$

$$(5) \quad \int_1^T R(t) dt = O(T),$$

and

$$(6) \quad \int_1^T R^2(t) dt = O(T(\log \log T)^4).$$

*Remarks.* In view of (4) it is seen that  $R(T)$  represents the error term in the mean square formula for  $\zeta(s)$  on the line  $\sigma = 1$ .

One often takes the lower limit of integration in (1) as zero. However, in our case ( $\sigma = 1$ ) this cannot be done, since  $\zeta(s)$  has a pole at  $s = 1$ . If in (3) we take some other positive number as the lower limit of integration, then obviously the value of  $R(T)$  will be changed by a constant only.

The method used in the proof of our theorem may be used to evaluate mean values of certain other zeta-functions on the line  $\sigma = 1$ . These will be dealt with elsewhere.

The upper bound in (4) contains information about the order of  $\zeta(1+iT)$ . Namely, from the general inequalities proved in [1] and [2] it may be deduced that

$$\zeta^k(1+iT) \ll \log^{2/3} T (\log \log T)^{1/3} \int_{T+\delta}^{T+\delta} |\zeta(1+it)|^k dt + 1$$

for any fixed integer  $k \geq 1$  and  $\delta = (\log \log T / \log T)^{2/3}$ . Hence we obtain

$$\zeta(1+iT) \ll \log^{2/3} T (\log \log T)^{1/3}.$$

This bound is close to the classical bound of I. M. Vinogradov (see Ch. 6 of [3])

$$\zeta(1 + iT) \ll \log^{2/3} T,$$

which for more than 30 years is the sharpest one, and follows from (19).

In view of (5) it seems plausible to conjecture that, as  $T \rightarrow \infty$ ,

$$\int_1^T R^2(t) dt \sim AT$$

for some  $A > 0$ .

*Acknowledgement.* We are very much indebted to the referee for his remarks concerning the proof of (4).

*Proof of the Theorem.* We start from the simplest approximate functional equation for  $\zeta(s)$  (see Th. 1.8 of [3]). Namely, for  $0 < \sigma_0 \leq \sigma \leq 2$ ,  $x \geq |t|/\pi$ ,  $s = \sigma + it$ , we have

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}).$$

For  $t \leq T$  this gives

$$(7) \quad \zeta(1 + it) = \sum_{n \leq T} n^{-1-it} + \frac{T^{-it}}{it} + O\left(\frac{1}{T}\right).$$

Since  $|z|^2 = z\bar{z}$  we obtain from (7)

$$(8) \quad \int_1^T |\zeta(1 + it)|^2 dt = \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right|^2 dt - 2\operatorname{Re} \left\{ \frac{1}{i} \int_1^T \sum_{n \leq T} \left(\frac{T}{n}\right)^{it} \frac{dt}{nt} \right\} \\ + O\left(\frac{1}{T} \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right| dt\right) + O\left(\frac{\log T}{T}\right).$$

The main terms in the asymptotic formula (3) come from the first integral on the right-hand side of (8). To see this one can use the well-known Montgomery-Vaughan theorem for mean values of Dirichlet polynomials (see [6], also [7] and Ch. 5 of [3]), which says that

$$(9) \quad \int_0^T \left| \sum_{n \leq N} a_n n^{-it} \right|^2 dt = T \sum_{n \leq N} |a_n|^2 + O\left(\sum_{n \leq N} n |a_n|^2\right)$$

for arbitrary complex numbers  $a_n$ ,  $T > 0$  and  $N \geq 1$ . Therefore by the Cauchy-Schwarz inequality and (9) we have

$$(10) \quad \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right| dt \leq T^{1/2} \left( \int_1^T \left| \sum_{n \leq T} n^{-1-it} \right|^2 dt \right)^{1/2} = O(T).$$

We also have

$$(11) \quad \int_1^T \sum_{n \leq T} \left( \frac{T}{n} \right)^{it} \frac{dt}{nt} = O(\log \log T).$$

To prove (11), let  $2 \leq H \leq T/2$  be a parameter. Then

$$\begin{aligned} & \int_1^T \sum_{n \leq T} \left( \frac{T}{n} \right)^{it} \frac{dt}{nt} = \sum_{n \leq T(1-1/H)} n^{-1} \int_1^T \left( \frac{T}{n} \right)^{it} \frac{dt}{t} \\ & + \sum_{T(1-1/H) < n \leq T} n^{-1} \int_1^T \left( \frac{T}{n} \right)^{it} \frac{dt}{t} = \sum_{n \leq T(1-1/H)} \frac{1}{n} \left\{ \frac{(T/n)^{it}}{it \log(T/n)} \Big|_1^T \right. \\ & \quad \left. + \int_1^T \frac{(T/n)^{it} dt}{it^2 \log(T/n)} \right\} + O \left( \sum_{T(1-1/H) < n \leq T} \frac{\log T}{n} \right) \\ & = O \left( \sum_{n \leq T(1-1/H)} \frac{1}{n \log(T/n)} \right) + O \left( \frac{\log T}{H} \right) \\ & \ll \int_1^{T(1-1/H)} \frac{dx}{x \log(T/x)} + \frac{\log T}{H} + 1 = \int_{(1-1/H)^{-1}}^T \frac{du}{u \log u} + \frac{\log T}{H} + 1 \\ & = \log \log T - \log \log \left( 1 - \frac{1}{H} \right)^{-1} + \frac{\log T}{H} + 1 \ll \log \log T \end{aligned}$$

for  $H = \log T$ . Thus from (8), (10) and (11) it follows that

$$(12) \quad \int_1^T \left| \zeta(1+iu) \right|^2 du = \int_1^T \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du + O(\log \log T).$$

Further we have

$$(13) \quad \int_1^T \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du = \int_0^T \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du - \int_0^1 \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du$$

and observe that by the Euler-Maclaurin summation formula (or by (7)) we have, for  $0 < u \leq 1$ ,

$$\sum_{n \leq T} n^{-1-iu} = \int_1^T x^{-1-iu} dx + \gamma + O(u) + O\left(\frac{1}{T}\right) = \frac{1 - T^{-iu}}{iu} \\ + \gamma + O(u) + O\left(\frac{1}{T}\right),$$

where  $\gamma = 0.5772157\dots$  is Euler's constant. Therefore

$$\int_0^1 \left| \sum_{n \leq T} n^{-1-iu} \right|^2 du = \int_0^1 \frac{|1 - T^{-iu}|^2}{u^2} du - 2\gamma \operatorname{Re} \left\{ i \int_0^1 \frac{1 - T^{-iu}}{u} du \right\} \\ (14) \quad + O(1) = 2 \int_0^1 \frac{1 - \cos(u \log T)}{u^2} du + 2\gamma \int_0^1 \frac{\sin(u \log T)}{u} du + O(1) \\ = 4 \int_0^1 \frac{\sin^2((u \log T)/2)}{u^2} du + O(1) = 2 \log T \int_0^{(\log T)/2} \left( \frac{\sin v}{v} \right)^2 dv + O(1) \\ = \pi \log T + O(1),$$

since

$$\int_0^{\infty} \left( \frac{\sin v}{v} \right)^2 dv = \frac{\pi}{2}.$$

In (12) – (14) we replace  $T$  by  $t$ , and suppose that  $T \leq t \leq 2T$ . From (7) we have

$$\sum_{T < n \leq t} n^{-1-it} = O\left(\frac{1}{T}\right).$$

Hence from the definition of  $R(T)$ , given by (3), it follows that for  $T \leq t \leq 2T$  we have

$$(15) \quad R(t) = \int_0^t \left( \left| \sum_{n \leq T} n^{-1-iu} \right|^2 - \zeta(2) \right) du + O(\log \log T).$$

This is the fundamental formula that will be used in the proof of (4) and (6). We start with the proof of (4), taking in (15)  $t = T$  and writing

$$\sum_{n \leq T} n^{-1-iu} = \sum_{n \leq N} + \sum_{N < n \leq 2N} + \sum_{2N < n \leq T} = \sum_1 + \sum_2 + \sum_3,$$

say, where

$$(16) \quad N = \exp(C \log^{2/3} T (\log \log T)^{1/3})$$

with a suitable constant  $C > 0$ . We have

$$(17) \quad \int_0^T \left| \sum_2 + \sum_3 \right|^2 du = \frac{1}{2} \int_{-T}^T \left| \sum_{N < n \leq T} n^{-1-iu} \right|^2 du \\ = T \sum_{N < n \leq T} n^{-2} + \sum_{N < m \neq n \leq T} \frac{(m/n)^{iT} - (m/n)^{-iT}}{2imn \log(m/n)}.$$

For the terms  $n < m \leq 2n$  in the second sum on the right-hand side of (17) we may put  $m = n + h$ , producing a sum of the form

$$(18) \quad \sum_{1 \leq h \leq T} \sum_{\max(N, h) < n \leq T} \frac{(1 + h/n)^{iT} - (1 + h/n)^{-iT}}{2i(n+h)n \log(1 + h/n)}.$$

To estimate the inner sum over  $n$  we shall apply the Vinogradov-Korobov estimate for exponential (zeta) sums. In its original form this says that

$$(19) \quad \sum_{M < m \leq M_1 \leq 2M} m^{it} \ll M \exp\left(-\frac{C \log^3 M}{\log^2 t}\right) \left(M_0 \leq M \leq \frac{1}{2}t, t \geq t_0\right),$$

and a proof (with  $C = 10^{-5}$ ) may be found in Ch. 6 of [3]. However, it is easily seen that the method of proof of (19) yields also

$$(20) \quad \sum_{M < m \leq M_1 \leq 2M} \left(1 + \frac{h}{m}\right)^{it} \ll M \exp\left(-\frac{C \log^3 M}{\log^2 t}\right) \left(M_0 \leq M_1 \leq \frac{1}{2}t, t \geq t_0\right)$$

with some absolute  $C > 0$ , provided that  $1 \leq h \leq m$ . Therefore (20) gives

$$\sum_{N' < n \leq N''} \left(1 + \frac{h}{n}\right)^{iT} \ll N' \log^{-4} T \quad (N \leq N' < N'' \leq 2N' \leq T),$$

provided that the constant  $C$  in (16) is sufficiently large. It follows by partial summation that the sum in (18) is  $\ll \log^{-2} T$ . Moreover the terms with  $m > 2n$  in (17) may be estimated directly by applying (19) to the sum over  $m$ , so that (17) becomes

$$(21) \quad \int_0^T \left| \sum_2 + \sum_3 \right|^2 du = T \sum_{N < n \leq T} n^{-2} + O(\log^{-2} T).$$

Similarly we find that

$$(22) \quad \int_{-T}^T \sum_{n \leq N} n^{-1-iu} \sum_{2N < m \leq T} m^{-1+iu} du \ll \log^{-2} T$$

for  $N$  given by (16) and  $C$  sufficiently large. Now by (15) we have

$$(23) \quad R(T) = \frac{1}{2} \int_{-T}^T \left| \sum_1 + \sum_2 + \sum_3 \right|^2 du - \zeta(2)T + O(\log \log T),$$

and we shall use the elementary identity

$$\begin{aligned} \left| \sum_1 + \sum_2 + \sum_3 \right|^2 &= \left| \sum_1 + \sum_2 \right|^2 - \left| \sum_2 \right|^2 + \left| \sum_2 + \sum_3 \right|^2 \\ &\quad + 2\operatorname{Re}(\sum_1 \bar{\sum}_3). \end{aligned}$$

By (9) the first two terms above contribute

$$T \sum_{n \leq N} n^{-2} + O(\log N)$$

to (23), while by (21) and (22) the contribution of the third and fourth term will be

$$T \sum_{N < n \leq T} n^{-2} + O(\log^{-2} T).$$

Therefore (23) gives

$$\begin{aligned} R(T) &= T \sum_{1 \leq n \leq T} n^{-2} - \zeta(2)T + O(\log N) + O(\log^{-2} T) \\ &= \zeta(2)T - \zeta(2)T + O(\log N) = O(\log^{2/3} T (\log \log T)^{1/3}), \end{aligned}$$

which proves (4). With a little more effort one could presumably remove the  $\log \log$  – factor in (4).

To prove (5) it suffices to prove

$$(24) \quad \int_T^{2T} R(t) dt = O(T),$$

and then to replace  $T$  by  $2^{-j}T$  and sum over  $j = 1, 2, \dots$ . For this purpose the error term in (15) is too large. Thus we use first (8), (10) and (14) to write, for  $T \leq t \leq 2T$ ,

$$\begin{aligned} R(t) &= \int_0^t \left( \left| \sum_{n \leq T} n^{-1-iu} \right|^2 - \zeta(2) \right) du - 2\operatorname{Re} \left\{ \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du}{inu} \right\} \\ &\quad + O(1), \end{aligned}$$

which is more precise than (15). We square out the first sum above and then integrate termwise, noting that interchanging  $m$  and  $n$  we have

$$\sum := \sum_{1 \leq m \neq n \leq T} \frac{1}{imn \log(m/n)} = \sum_{1 \leq m \neq n \leq T} \frac{1}{inm \log(n/m)} = -\sum,$$

hence  $\sum = 0$ . This gives, for  $T \leq t \leq 2T$ ,

$$(25) \quad R(t) = \sum_{1 \leq m \neq n \leq T} \frac{(m/n)^{it}}{imn \log(m/n)} - 2 \operatorname{Re} \left\{ \frac{1}{i} \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du}{un} \right\} + O(1).$$

Then we integrate (25) to obtain

$$\int_T^{2T} R(t) dt = \sum_{1 \leq m \neq n \leq T} \frac{(m/n)^{iT} - (m/n)^{2iT}}{mn \log^2(m/n)} - 2 \operatorname{Re} \left\{ \frac{1}{i} \int_T^{2T} \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} \right\} + O(T).$$

But the modulus of the double sum above does not exceed

$$\begin{aligned} & \sum_{1 \leq n < m \leq T} \frac{4}{mn \log^2(m/n)} = \sum_{1 \leq n < m \leq T, n < m/2} + \sum_{1 \leq n < m \leq T, n \geq m/2} \\ & \ll \log^2 T + \sum_{n \leq T} \sum_{n < m \leq 2n} \frac{1}{m^2 \log^2(m/n)} \ll \log^2 T + \sum_{n \leq T} \sum_{n < m \leq 2n} \frac{1}{(m-n)^2} \\ & \ll T, \end{aligned}$$

and it remains to estimate

$$\begin{aligned} I &:= \int_T^{2T} \int_1^t \sum_{n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} = \int_T^{2T} \int_1^{\log T} \sum_{n \leq T/2} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} \\ &+ \int_T^{2T} \int_{\log T}^t \sum_{n \leq T/2} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} + \int_T^{2T} \int_1^t \sum_{T/2 < n \leq T} \left( \frac{t}{n} \right)^{iu} \frac{du dt}{nu} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say. We have

$$\begin{aligned} I_1 &= \sum_{n \leq T/2} \frac{1}{n} \int_1^{\log T} \left( \int_T^{2T} \left( \frac{t}{n} \right)^{iu} \frac{du}{u} \right) dt \\ &= \sum_{n \leq T/2} \frac{1}{n} \int_1^{\log T} \left\{ \frac{(t/n)^{iu}}{iu \log(t/n)} \Big|_T^{2T} + \int_T^{2T} \frac{(t/n)^{iu}}{iu^2 \log(t/n)} du \right\} dt \\ &\ll \frac{\log T}{T} \sum_{n \leq T/2} \frac{1}{n \log(T/n)} \ll \frac{\log^2 T}{T}, \end{aligned}$$



$$\begin{aligned}
 I_2 &= \sum_{n \leq T/2} \frac{1}{n} \int_T^{2T} \left\{ \frac{(t/n)^{iu}}{iu \log(t/n)} \Big|_{\log T}^t + \int_{\log T}^t \frac{(t/n)^{iu}}{iu^2 \log(t/n)} du \right\} dt \\
 &\ll \sum_{n \leq T/2} \frac{T}{n \log(T/n) \log T} \ll T, \\
 I_3 &= \sum_{T/2 < n \leq T} \frac{1}{n} \int_1^{2T} \left( \int_{\max(u, T)}^{2T} \left(\frac{t}{n}\right)^{iu} dt \right) \frac{du}{u} \\
 &= \sum_{T/2 < n \leq T} \int_1^{2T} n^{-1-iu} \frac{(2T)^{iu+1} - (\max(u, T))^{iu+1}}{u(iu+1)} du \\
 &\ll T \sum_{T/2 < n \leq T} \frac{1}{n} \ll T,
 \end{aligned}$$

and (24) follows.

It remains yet to prove (6), which follows from

$$(26) \quad \int_T^{2T} R^2(t) dt = O(T(\log \log T)^4).$$

We use (25) and (11) to obtain

$$\begin{aligned}
 R^2(t) &\leq \sum_{a \neq b \leq T, c \neq d \leq T} \frac{-2 \left(\frac{ad}{bc}\right)^{it}}{abcd \log(a/b) \log(d/c)} + O((\log \log T)^2) \\
 &= \sum_{ad=bc} + \sum_{0 < |\log(ad/bc)| \leq T^{-1} \log^4 T} + \sum_{|\log(ad/bc)| > T^{-1} \log^4 T} \\
 &\quad + O((\log \log T)^2) = S_0 + S_1 + S_2 + O((\log \log T)^2),
 \end{aligned}$$

say. On integrating we obtain

$$\begin{aligned}
 &\int_T^{2T} S_2 dt \\
 &= 2 \sum_{a \neq b \leq T, c \neq d \leq T, |\log(ad/bc)| > T^{-1} \log^4 T} \frac{\left(\frac{ad}{bc}\right)^{iT} - \left(\frac{ad}{bc}\right)^{2iT}}{iabcd \log\left(\frac{a}{b}\right) \log\left(\frac{d}{c}\right) \log\left(\frac{ad}{bc}\right)} \\
 &\ll \frac{T}{\log^4 T} \left( \sum_{a \neq b \leq T} \frac{1}{ab |\log(a/b)|} \right)^2 \ll T,
 \end{aligned}$$

since an elementary argument easily gives

$$\sum_{a \neq b \leq T} \frac{1}{ab |\log(a/b)|} \ll \log^2 T.$$

The remaining sums  $S_0$  and  $S_1$  are not integrated, but it is sufficient to show that

$$(27) \quad S_0 \ll 1, S_1 \ll (\log \log T)^4.$$

We have

$$\begin{aligned} S_0 &= \sum_{ad=bc, a \neq b \leq T, c \neq d \leq T} \frac{1}{abcd \log(a/b) \log(d/c)} \\ &= \sum_k \frac{1}{k^2} \sum_{ad=bc=k, a \neq b \leq T} \frac{1}{\log^2(a/b)} = \sum_{a \neq b \leq T} \frac{1}{\log^2(a/b)} \sum_{a|k, b|k} \frac{1}{k^2} \\ &\ll \sum_{a \neq b \leq T} \frac{1}{[a, b]^2 \log^2(a/b)} \leq \sum_{j=1}^{\infty} \sum_{a \neq b, (a, b)=j} \frac{1}{\log^2(a/b)} \left( \frac{j}{ab} \right)^2 \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{a' \geq 1, b' \geq 1, a' \neq b'} \frac{1}{\log^2(a'/b') (a'b')^2} = O(1). \end{aligned}$$

The proof of the other bound in (27) is also elementary, but somewhat more involved. Write

$$S_1 = S_3 + S_4,$$

where in  $S_3$  summation is over  $a, b, c, d$  such that  $1 \leq a \neq b \leq T$ ,

$$1 \leq c \neq d \leq T, |\log a/b| \geq T^{-1/4}, |\log c/d| \geq T^{-1/4},$$

$$0 < \left| \log \frac{ad}{bc} \right| \leq T^{-1} \log^4 T,$$

and in  $S_4$  over the remaining values of  $a, b, c, d$ . Thus

$$\begin{aligned} S_3 &\ll T^{1/2} \sum_{a, b, c, d \leq T; |\log(ad/bc)| \leq T^{-1} \log^4 T} \frac{1}{abcd} \\ &\ll T^{1/2} \sum_{|\log(k/l)| \leq T^{-1} \log^4 T; k \neq l \leq T^2} \frac{d(k)d(l)}{kl}. \end{aligned}$$

Now observe that if  $k \neq l \geq 1$  are integers, then

$$(28) \quad \left| \log \frac{k}{l} \right| \geq \log \frac{l+1}{l} \geq \frac{1}{2l},$$

so that

$$l \geq \frac{1}{2} T \log^{-4} T, \quad \frac{|k-l|}{l} \ll \left| \log \left( 1 + \frac{k-l}{l} \right) \right| \ll T^{-1} \log^4 T.$$

Thus

$$S_3 \ll T^{1/2+\varepsilon} \sum_{T/(2\log^4 T) \leq l \leq T^2} \frac{1}{l^2} \left( \frac{\log^4 T}{T} + 1 \right) \ll T^{-1/2+2\varepsilon} \ll 1$$

for  $0 < \varepsilon \leq 1/4$ . In  $S_4$  we have either  $|\log(a/b)| \leq T^{-1/4}$  or  $|\log(d/c)| \leq T^{-1/4}$ . If the former holds, then

$$\left| \log \frac{d}{c} \right| = \left| \log \frac{ad}{bc} - \log \frac{a}{b} \right| \leq 2T^{-1/4},$$

so that in  $S_4$  we have  $|\log(a/b)| \leq 2T^{-1/4}$ ,  $|\log(d/c)| \leq 2T^{-1/4}$ , and also  $a \ll b \ll a, c \ll d \ll c$ . Setting  $a = b + j_1, c = d + j_2$ , we have that  $j_1 \ll bT^{-1/4}, j_2 \ll dT^{-1/4}$ , and

$$(29) \quad \left| \frac{j_1}{b} - \frac{j_2}{d} \right| \ll \frac{\log^4 T}{T},$$

since

$$\begin{aligned} \left| \frac{a}{b} - \frac{c}{d} \right| &= \frac{|ad - bc|}{bd} \ll \frac{|ad - bc|}{bc} \ll \left| \log \left( 1 + \frac{ad - bc}{bc} \right) \right| \\ &= \left| \log \frac{ad}{bc} \right| \ll \frac{\log^4 T}{T}. \end{aligned}$$

Hence

$$S_4 \ll \sum_{b, d, j_1, j_2}^* \frac{1}{bdj_1j_2},$$

where  $\sum^*$  denotes summation with the conditions  $b, d \leq T; j_1 \ll bT^{-1/4}, j_2 \ll dT^{-1/4}$  and (29) satisfied. We have

$$S_4 = S_5 + S_6,$$

where trivially

$$S_5 = \sum_{T \log^{-10} T \leq b, d, \leq T; j_1, j_2 \leq \log^{20} T} \frac{1}{bdj_1j_2} \ll (\log \log T)^4,$$

and in  $S_6$  we have (by symmetry) either

$$\text{i) } j_2 \geq \log^{20} T$$

or

$$\text{ii) } d \leq T \log^{-10} T.$$

From (29) it follows that

$$(30) \quad d - \frac{bj_2}{j_1} \ll \frac{bd \log^4 T}{j_1 T}.$$

Suppose that i) holds. Then  $d - bj_2 j_1^{-1} \ll j_1^{-1} b \log^4 T$  from (30), hence the corresponding part of  $S_6$  is

$$\begin{aligned} &\ll \sum_{b \leq T} \frac{1}{b} \sum_{j_1 \ll bT^{-1/4}} \frac{1}{j_1} \sum_{j_2 \geq \log^{20} T} \frac{1}{j_2} \sum_{|d - bj_2 j_1^{-1}| \leq Cbj_1^{-1} \log^4 T} \frac{1}{d} \\ &\ll \sum_{b \leq T} \frac{1}{b} \sum_{j_1 \ll bT^{-1/4}} \frac{1}{j_1} \sum_{j_2 \geq \log^{20} T} j_2^{-2} \log^4 T \ll \log^{-4} T. \end{aligned}$$

If ii) holds, then (30) gives

$$d - \frac{bj_2}{j_1} \ll \frac{b}{j_1} \log^{-6} T,$$

and the corresponding contribution to  $S_6$  will be again  $\ll \log^{-4} T$ . This proves (27) and completes the proof of the Theorem.