2. Action of σ_{n+1} on $\rho_k(\Sigma_k)$

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In this section, we show how the n usual generators $(0,1), \ldots, (n-1,n)$ of the group σ_{n+1} act on the subset $\rho_k(\Sigma_k)$ of $(\mathbf{Z}/k\mathbf{Z})^{n-1}$.

LEMMA 2.1. *Let* $(\alpha_1, .., \alpha_{n-1}) \in \rho_k(\Sigma_k)$. *Then*

$$(0,1)\cdot(\alpha_1,\alpha_2,..,\alpha_i,..,\alpha_{n-1}) = ((1-\alpha_1-\alpha_2-..-\alpha_{n-1}),\alpha_2, .., \alpha_i,..,\alpha_{n-1})$$

$$(i,i+1)\cdot(\alpha_1,..,\alpha_i,\alpha_{i+1},..,\alpha_{n-1}) = (\alpha_1,..,\alpha_{i+1},\alpha_i,..\alpha_{n-1}), \quad 1 \leqslant i \leqslant n-2$$

$$(n-1,n)\cdot(\alpha_1,..,\alpha_i,..,\alpha_{n-2},\alpha_{n-1}) = (-\alpha_1\alpha_{n-1}^{-1},..,-\alpha_i\alpha_{n-1}^{-1},..,-\alpha_{n-2}\alpha_{n-1}^{-1},\alpha_{n-1}^{-1})$$

In particular, if $(\alpha_1,..,\alpha_{n-1}) \in \rho_k(\Sigma_k)$, *then* α_{n-1} *is invertible* (mod k).

Proof. Let us show the first equality. Let $(\alpha_1,..,\alpha_{n-1}) \in \rho_k(\Sigma_k)$. Consider the numerated simplex S_v with vertices

$$v_0 = 0, v_i = e_i, 1 \leqslant i \leqslant n-1, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i \tag{2.2}$$

(a_i representant of the class α_i).

Let us identify $\mathbf{R}^n$ with the points of the hyperplane H of $\mathbf{R}^{n+1}$ defined by $x_{n+1} = 1$. By identifying the elements of $\mathbf{R}^n$ with vector columns we see that

$$0 \mapsto \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad e_i \mapsto \begin{pmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

where 0 is the origin in $\mathbf{R}^n$ and $\{e_1, \ldots, e_n\}$ is the usual basis of $\mathbf{R}^n$ and the 1 in the second matrix is where you think it should be, namely at the i-th row. There exists a natural injection of the group $\mathrm{Aff}(\mathbf{Z}^n)$ in the subgroup of $GL_{n+1}(\mathbf{Z})$ which preserves H. It is easy to check that the following matrix $M \in GL_{n+1}(\mathbf{Z})$ exchanges v_0 and v_1, preserves v_i for $2 \leqslant i < n$ and sends v_n to the element $ke_n + (1 - \sum_{i=1}^{n-1} a_i)e_1 + \sum_{i=2}^{n-1} a_i e_i$:

$$M = \begin{pmatrix} -1 & -1 & -1 & .. & -1 & 0 & 1 \\ 0 & 1 & 0 & .. & 0 & 0 & 0 \\ 0 & 0 & 1 & .. & 0 & 0 & 0 \\ .. & . & . & .. & . & . & . \\ 0 & 0 & 0 & .. & 1 & 0 & 0 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 \\ 0 & 0 & 0 & .. & 0 & 0 & 1 \end{pmatrix}.$$

The calculations for the transposition $(i, i+1)$ are immediate if $1 \leqslant i \leqslant n-2$.

Finally, let us consider the last equality: We take again the simplex S_ν with vertices as in (2.2). Since S_ν is small-faced, there exists a simplex S'_μ, $(n-1)$ integers $b_1, .., b_{n-1}$ and an element $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that the vertices of S'_μ are

$$v'_0 = 0, v'_i = e_i, 1 \leqslant i \leqslant n-1, v'_n = ke_n + \sum_{i=1}^{n-1} b_i e_i$$

and such that $g(v_i) = v'_i$ for $0 \leqslant i \leqslant n-2$, $g(v_{n-1}) = v'_n$ and $g(v_n) = v'_{n-1}$. Since $v_0 = v'_0 = 0$ we have $g(0) = 0$ and g is in fact in $GL_n(\mathbf{Z})$. The matrix of g with respect to the standard basis is

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & b_2 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & 0 \\ 0 & 1 & \dots & 0 & a_2 & 0 \\ . & . & \dots & . & . & . \\ 0 & 0 & \dots & 1 & a_{n-2} & 0 \\ 0 & 0 & \dots & 0 & a_{n-1} & 1 \\ 0 & 0 & \dots & 0 & k & 0 \end{pmatrix}^{-1}$$

this gives

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & b_2 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix} \frac{1}{k} \begin{pmatrix} k & 0 & 0 & \dots & 0 & 0 & -a_1 \\ 0 & k & 0 & \dots & 0 & 0 & -a_2 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & k & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & k & -a_{n-1} \end{pmatrix}.$$

But since $g \in GL_n(\mathbf{Z})$ this implies that

$$-a_i - b_i a_{n-1} \equiv 0 \pmod{k} \qquad 1 \leqslant i \leqslant n-2\,.$$

Let us now suppose that a_{n-1} is not invertible (mod k). Then there exists some prime p dividing both k and a_{n-1}. But then the prime p divides a_i too for every i. So p divides all coefficients of the vector $v_n - v_0$ which is an edge of S_ν. But then S_ν is not small-faced which contradicts the fact that $(\alpha_1, .., \alpha_{n-1}) \in \rho_k(\Sigma_k)$. So we have proved that a_{n-1} is invertible (mod k). And the b_i's satisfy

$$b_i \equiv -a_i a_{n-1}^{-1} \pmod{k} \quad 1 \leqslant i \leqslant n-2 \quad \text{and} \quad b_{n-1} \equiv a_{n-1}^{-1} \pmod{k}\,.$$

This proves the last equality. $\square$