3. Eigenfunctions

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$$= f(1)g(0) + \sum_{n=0}^{\infty} qf(n)g(n+1)q^{-(n+1)} + \sum_{n=2}^{\infty} f(n)g(n-1)q^{-(n-1)}$$

$$= f(1)g(0) + f(0)g(1) + \sum_{n=1}^{\infty} f(n)g(n+1)q^{-n}$$

$$+ q\sum_{n=1}^{\infty} f(n)g(n-1)q^{-n} - f(1)g(0)$$

$$= f(0)g(1) + \sum_{n=0}^{\infty} (f(n)qg(n-1) + f(n)g(n+1))q^{-n} = <f, T\bar{g}> .$$

3. Eigenfunctions

An automorphic eigenfunction of T on X with eigenvalue λ is a function on F that satisfies

$$\lambda f(0) = (q+1)f(1) ,$$
$$\lambda f(n) = qf(n-1) + f(n+1) , \quad n \geqslant 1 .$$

If we write $u(n) = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$ and normalize $u(0) = \begin{pmatrix} \lambda \\ q+1 \end{pmatrix}$, we obtain the recursion

$$u(n) = A^n u(0)$$

with

$$A = \begin{pmatrix} \lambda & -q \\ 1 & 0 \end{pmatrix} .$$

Let $x_1, x_2 = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4q})$ be the characteristic roots of A and assume that $x_1 \neq x_2$, i.e., that $\lambda \neq \pm 2\sqrt{q}$. Solving the recursion we get

PROPOSITION 3.1. *The eigenfunctions on* F *with eigenvalue* λ *are the multiples of the function*

$$(4) \quad f_\lambda(n) = \begin{cases} \dfrac{1}{x_1 - x_2}(\lambda(x_1^n - x_2^n) - q(q+1)(x_1^{n-1} - x_2^{n-1})) , & \text{if } n \geqslant 1 \\ q+1 \quad \text{if} \quad n = 0 . \end{cases}$$

Example 3.2. If $\lambda = q + 1$ then $x_1 = q, x_2 = 1$ and

$$f_{q+1}(n) \equiv q + 1 \ ,$$

generating the space of constant functions. If $\lambda = -(q + 1)$, then

$$f_{-(q+1)}(n) = (-1)^n(q + 1) \ .$$

Example 3.3. When $\lambda = 2\sqrt{q}$ we can solve directly to get

$$f_{2\sqrt{q}}(n) = (q + 1 - (q - 1)n)q^{\frac{n}{2}},$$

and similarly

$$f_{-2\sqrt{q}}(n) = (-1)^n(q + 1 - (q - 1)n)q^{\frac{n}{2}} \ .$$

Remark 3.4. Since our tree is bipartite, we expect f_λ to be related to $f_{-\lambda}$ by a factor of $(-1)^n$ (compare [B, § 8]). This can be seen from (4).

PROPOSITION 3.5. *The only eigenvalues λ with $|\lambda| > 2\sqrt{q}$ for which f_λ is in $L^2(F)$ are $\lambda = \pm (q + 1)$.*

Proof. Recalling (2) we see that if $f_\lambda \in L^2(F)$ then

$$f_\lambda(n) = o(q^{\frac{n}{2}}) \quad \text{as} \quad n \to \infty \ .$$

Now

$$(x_1 - x_2)f_\lambda(n) = x_1^{n-1}(\lambda x_1 - q(q + 1)) - x_2^{n-1}(\lambda x_2 - q(q + 1)) \ .$$

Assuming with no loss of generality that $|x_1| > \sqrt{q}$, $|x_2| = \dfrac{q}{|x_1|} < \sqrt{q}$, then $x_2^{n-1}(\lambda x_2 - q(q + 1)) = o(q^{\frac{n}{2}})$, so that we must have

$$\lambda x_1 - q(q + 1) = 0 \ ,$$

i.e., $\lambda = \pm (q + 1)$. Conversely, f_{q+1} and $f_{-(q+1)}$ are clearly in $L^2(F)$.

We turn our attention to λ with $|\lambda| < 2\sqrt{q}$. Then $x_2 = \bar{x}_1$, $|x_1| = \sqrt{q}$ and we let $x_1 = \sqrt{q}e^{i\theta}$. Then $\lambda = 2\sqrt{q}\cos\theta, 0 < \theta < \pi$. We renormalize and define

$$\tilde{f}_\theta = \frac{x_1 - x_2}{2\sqrt{q}} f_{2\sqrt{q}\cos\theta} \ .$$

Then, for $n \geqslant 1$,

$$\begin{aligned}\tilde{f}_\theta(n) &= q^{\frac{n}{2}} i((q+1)\sin\theta\cos(n\theta) - (q-1)\cos\theta\sin(n\theta)) \\ &= q^{\frac{n}{2}} i(\sin((n+1)\theta) - q\sin((n-1)\theta)) \,,\end{aligned} \tag{5}$$

and

$$\tilde{f}_\theta(0) = (q+1)\, i\sin\theta \,.$$

PROPOSITION 3.6. *The functions* $\tilde{f}_\theta, 0 < \theta < \pi$, *are not in* $L^2(F)$.

Proof. It is sufficient to show that

$$(q+1)\sin\theta\cos(n\theta) - (q-1)\cos\theta\sin(n\theta) \not\to 0 \quad \text{as} \quad n \to \infty \,.$$

This is the dot product of the two vectors

$$v_1 = ((q+1)\sin\theta, -(q-1)\cos\theta) \quad \text{and} \quad v_2 = (\cos(n\theta), \sin(n\theta)) \,.$$

Since v_2 is not a constant function of n, we see that cosine of the angle between v_1 and v_2 is bounded away from 0 for arbitrarily large n.

Combining Propositions 3.5 and 3.6 we conclude

COROLLARY 3.7. *The discrete spectrum of* T *consists of the numbers* $\pm(q+1)$, *whose corresponding eigenfunctions (given in Example 3.1) span two one-dimensional eigenspaces of* $L^2(F)$.

Unlike the typical f_λ with $|\lambda| > 2\sqrt{q}$, those with $|\lambda| < 2\sqrt{q}$ satisfy

$$f_\lambda = O(q^{\frac{n}{2}}) \,.$$

Our goal now is to show that these are approximate eigenfunctions that can be used to completely decompose $L^2(F)$.

4. CONTINUOUS SPECTRA

We wish to embed $L^2([0,\pi])$ with an appropriate measure into $L^2(F)$. To this end, let $\psi \in L^2([0,\pi])$ and $\tilde{f}_\theta(n)$ be extended as odd functions of $\theta \in [-\pi,\pi]$, and define

$$F_\psi(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(\theta)\tilde{f}_\theta(n)\,d\theta \,.$$