Introduction

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ON THE DIFFEOMORPHISM GROUPS OF CERTAIN ALGEBRAIC SURFACES

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INTRODUCTION

It has become clear in the last years that there is an essential difference between the topology and the differential topology of 4-manifolds. The new understanding is based on fundamental ideas and results due to M. Freedman and S.K. Donaldson. The difference between the two categories becomes particularly apparent for the underlying 4-manifolds of complex algebraic surfaces. There are e.g. algebraic surfaces whose underlying topological manifolds carry infinitely many differentiable structures.

The object of this paper is to study in an analogous way the automorphisms of the two categories, i.e. the symmetry groups of 4-manifolds. For the topological category this is well understood. By the result of Freedman [F], simply-connected closed oriented topological 4-manifolds are classified up to homeomorphism by the isomorphism class of a lattice and the Kirby-Siebenmann invariant in $\mathbf{Z}/2$. If X is such a manifold, then the lattice in question is $L = (H_2(X, \mathbf{Z}), q_X)$, where q_X is the quadratic intersection form of X. An orientation preserving homeomorphism of X induces an isometry of L, and isotopic homeomorphisms induce the same isometry. Therefore there is a natural homomorphism ψ_{top} from the component group $\text{Top}_+(X)$ of the group of orientation preserving homeomorphisms of X to the orthogonal group $O(L)$ of the lattice L. F. Quinn has shown that this homomorphism is in fact an isomorphism [Q].

Now assume in addition that X is a differentiable manifold. Then one considers the component group $\text{Diff}_+(X)$ of the group of orientation preserving diffeomorphisms of X, and again there is a natural homomorphism $\psi\colon \text{Diff}_+(X) \to O(L)$. There are some classical results related to ψ which have been obtained via traditional (surgery) methods: It was first observed by C.T.C. Wall [W] that ψ is surjective for $S^2 \times S^2$, for the connected sum $\mathbf{P}^2 \# \overline{\mathbf{P}^2}$ of the complex projective plane $\mathbf{P}^2$ with another copy $\overline{\mathbf{P}^2}$ with reversed orientation, and for the connected sum of several copies of $\mathbf{P}^2$. Wall has also shown that ψ is surjective if X is of the form $X = Y \# S^2 \times S^2$ where Y is a manifold whose quadratic form q_Y is indefinite or has rank $\leqslant 8$ [W]. M. Kreck has proved that the image of ψ is isomorphic to the group $\widetilde{\mathrm{Diff}}_+(X)$ of pseudo-isotopy classes of orientation preserving diffeomorphisms [Kr].

In general it is therefore a difficult problem to determine the image of ψ.

With the new methods provided by Donaldson, the following results were obtained. Suppose that X is a complex algebraic surface with canonical class k_X. For a Dolgachev surface X, R. Friedman and J. Morgan have shown that the image of ψ is of finite index in the subgroup of $O(L)$ consisting of isometries preserving $\{\pm k_X\}$; this subgroup itself is of infinite index in $O(L)$ [FM]. Recently Donaldson has determined the image of ψ for a K3 surface X; he has identified it with a certain subgroup of index 2 in $O(L)$ [D].

The main result of this paper describes the image of ψ for other types of algebraic surfaces. For the precise formulation of the result we refer to §4. The proof is inspired by Donaldson's proof. It has three main parts.

First we exhibit a large subgroup of $\psi(\mathrm{Diff}_+(X))$. This is the monodromy group of a smooth family of surfaces containing X as a fibre. In many cases we can determine the monodromy group of a suitable family. Moreover a diffeomorphism can be constructed using complex conjugation. §1 deals with these topics.

The next ingredient is the C^∞-invariance of the canonical class. This is proved for certain algebraic surfaces using Donaldson's $SO(3)$-polynomial invariants and the results of [FMM]. This is the subject of §2.

Finally we show (in §3) that $-1 \in O(L)$ is not induced by an orientation preserving diffeomorphism if X is an algebraic surface with odd geometric genus.

The main new ingredient is the use of Donaldson's $SO(3)$-invariants, where we rely on the recent nontriviality result of K. Zuo [Z].

We conclude the paper with two applications. The first one concerns the problem of representing homology classes in $H_2(X, \mathbf{Z})$ by differentiably embedded 2-spheres. For a second application we consider homeomorphic surfaces X and X' to which our results apply and for which we can determine the images of the corresponding homomorphisms ψ. We show that if the divisibilities of the canonical classes of X and X' in integral cohomology are different (and hence the surfaces are not diffeomorphic), then the corresponding images are non-conjugate subgroups of $O(L)$, and vice versa. Finally we give an example of two such surfaces.

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1. Construction of diffeomorphisms

Let X be a simply connected smooth compact complex algebraic surface with quadratic intersection form $q_X: H_2(X, \mathbf{Z}) \to \mathbf{Z}$ and canonical class $k_X \in H^2(X, \mathbf{Z}) = \operatorname{Hom}(H_2(X, \mathbf{Z}), \mathbf{Z})$. The second homology group $H_2(X, \mathbf{Z})$ endowed with the quadratic form q_X forms a lattice which we denote by L. Let $O(L)$ be the corresponding group of isometries.

We introduce some notation: Let $b_2^+(X)$ denote the dimension of any maximal subspace of $L_{\mathbf{R}} = L \otimes \mathbf{R}$ on which q_X is positive definite. Note that $b_2^+(X) = 2p_g(X) + 1$ where $p_g(X)$ is the geometric genus of X. The set of all oriented maximal positive definite subspaces of $L_{\mathbf{R}}$ forms an open subset Ω of the Grassmannian $G^{\mathrm{or}}(b_2^+(X), L_{\mathbf{R}})$ of oriented $b_2^+(X)$-dimensional subspaces of the vector space $L_{\mathbf{R}}$. It has two components if q_X is indefinite. We define $O'(L)$ to be the subgroup of $O(L)$ consisting of those automorphisms which leave each component of Ω invariant. (For an equivalent definition of $O'(L)$ see [E2, 4.1].) Let $O_k(L)$ be the subgroup of $O(L)$ consisting of automorphisms preserving $k = k_X$. Finally we define $O'_k(L) := O_k(L) \cap O'(L)$.

An important subset of $\operatorname{Diff}_+(X)$ is the set of classes of diffeomorphisms obtained by monodromy transformations of a smooth family containing X as a fibre. By a *smooth family* we mean a smooth (in the analytic category) proper holomorphic mapping $\pi: \mathscr{X} \to T$ of connected complex spaces $\mathscr{X}$ and T; π is the projection of a locally trivial differentiable fibre bundle, so that for a point $t_0 \in T$ with $X = \pi^{-1}(t_0)$ there is a *monodromy representation* $\rho: \pi_1(T, t_0) \to \operatorname{Diff}_+(X)$. The image Γ of $\psi \circ \rho$ in $O(L)$ is called the *monodromy group* of the smooth family. The monodromy group preserves k_X. It also preserves the components of Ω: to see this consider a loop τ representing an element in $\pi_1(T, t_0)$ and let $g_t: X_{t_0} = X \to X_t$ be the diffeomorphisms corresponding to $t \in \tau$. The mapping $t \mapsto (g_t)_*(\alpha) \in G^{\mathrm{or}}(b_2^+(X), L_{\mathbf{R}})$ is continuous for every $\alpha \in \Omega$, hence $\Gamma \subset O'_k(L)$.

For certain algebraic surfaces there exist smooth families whose monodromy group is the whole group $O'_k(L)$. This is summarized in the