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ON THE AVERAGE BEHAVIOUR OF THE LARGEST DIVISOR OF n PRIME TO A FIXED INTEGER k

by Y.-F. S. PÉTERMANN

RÉSUMÉ. On étudie le comportement de la fonction bornée $h_k(x) := x^{-1}E_k(x)$, où $E_k(x) := \sum_{n \leqslant x} \delta_k(n) - (k/2\sigma(k))x^2$ est le terme irrégulier du comportement asymptotique moyen de $\delta_k(n)$, le plus grand diviseur de n premier à k (et où l'on peut sans perte supposer que k est sans facteur carré). On s'intéresse plus particulièrement aux nombres I_k et S_k, les lim inf et lim sup de $h_k(x)$ (lorsque $x \to \infty$), dont les valeurs exactes ne sont connues que si $k = 1$ ou si k est un nombre premier (Joshi et Vaidya [JV]). En établissant l'existence et la symétrie de la fonction de répartition de $h_k(n)$ (au sens de Wintner), on simplifie le problème en démontrant que $I_k = -S_k$. Puis, pour tous les k non premiers et sans facteur carré, on améliore explicitement l'estimation $S_k \geqslant k/\sigma(k)$ (de Herzog et Maxsein [HM], et indépendamment Adhikari, Balasubramanian et Sankaranarayanan [ABS]).

0. INTRODUCTION AND STATEMENT OF THE RESULTS

For a fixed natural number k we denote by $\delta_k(n)$ the largest divisor of n which is prime to k. If κ is the squarefree core of k we have $\delta_k(n) = \delta_\kappa(n)$, and we shall assume from now on that k is squarefree. We define the associated error term

$$E_k(x) := \sum_{n \leqslant x} \delta_k(n) - \frac{k}{2\sigma(k)} x^2 , \tag{0.1}$$

where σ is the sum-of-divisors function. The behaviour of $E_k(x)$ has been investigated in [Su], [JV], [HM], [ABS], [AB], and very recently in [A]. It is known that [JV]

$$E_k(x) = O(x) \tag{0.2}$$

and that [JV, HM, ABS] [1])

(0.3) $$E_k(x) = \Omega_{\pm}(x) .$$

However, the exact values of the lim sup and lim inf of $E_k(x)/x$ are not known, except in the special case where k is a prime p (and of course when $k = 1$); we have [JV]

(0.4) $$\limsup_{x \to \infty} \frac{E_p(x)}{x} = \frac{p}{p+1} \quad \text{and} \quad \liminf_{x \to \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1} .$$

Let us from now on use the notation

(0.5) $$S_k := \limsup_{x \to \infty} \frac{E_k(x)}{x} \quad \text{and} \quad I_k := \liminf_{x \to \infty} \frac{E_k(x)}{x} .$$

When the number $\omega(k)$ of (distinct) prime divisors of k exceeds 1, the best estimates known so far are on the one hand [HM, ABS]

(0.6) $$S_k \geqslant \frac{k}{\sigma(k)} \quad \text{and} \quad I_k \leqslant -\frac{k}{\sigma(k)} ,$$

and on the other hand [A]

(0.7) $$S_k \leqslant C(k) \quad \text{and} \quad I_k \geqslant -C(k)$$

where, if $k = p_1 p_2 \ldots p_r$ $(p_1 < p_2 < \ldots < p_r)$,

$$C(k) := \frac{p_1}{p_1 + 1} 2^{r-1} - \sum_{j=2}^{r} \frac{p_1 p_2 \ldots p_{j-1}}{(p_1 + 1)(p_2 + 1) \ldots (p_j + 1)} 2^{r-j} .$$

The purpose of this note is to improve on the estimates (0.6) for all k with $\omega(k) \geqslant 2$. As a preliminary we simplify the study of $E_k(x)$; in Section 1 we prove

THEOREM 1. *The function*

(0.8) $$h(x) = h_k(x) := \frac{E_k(x)}{x}$$

[1]) The notation in (0.3) means that there are two unbounded positive sequences $\{x_i^+\}$ and $\{x_i^-\}$ $(i = 1, 2, \ldots)$, and two strictly positive constants C^+ and C^-, such that the inequalities $E_k(x_i^+) \geqslant C^+ x_i^+$ and $E_k(x_i^-) \leqslant -C^- x_i^-$ hold for each $i = 1, 2, \ldots$.

possesses an asymptotic distribution function which is symmetric (and of bounded support). Moreover we have

$$I_k = - S_k . \tag{0.9}$$

Then we obtain in Section 2 a lower bound for S_k in the case where $k = pq$ ($p < q$ primes) which implies in particular

THEOREM 2. *For* $k = 2q \geqslant 6$ *where* q *is a prime we have*

$$S_k \geqslant \frac{q - \frac{1}{3}}{q + 1} = \frac{k}{\sigma(k)} + \frac{q - 1}{3(q + 1)} . \tag{0.10}$$

And finally in Section 3 we show

THEOREM 3. *Let* $k = p_1 p_2 \dots p_r$, *where* $p_1 < p_2 < \dots < p_r$ *are primes and* $r \geqslant 2$, *and let* N *be the positive integer such that*

$$\begin{aligned} f_r(p_2, \dots, p_r) : &= \left(\frac{\sigma(k/p_1}{k/p_1} - 1 \right)^{-1} \\ &\in \begin{cases} (0, p_1^2 - 1) & (N = 1) \\ [p_1^N - 1, p_1^{N+1} - 1) & (N = 2, 3, \dots) . \end{cases} \end{aligned} \tag{0.11}$$

Then, except possibly in the case where $r = 2$, $p_1 = 2$ *and* $p_2 = 2^N - 1$, *we have*

$$\begin{aligned} S_k &\geqslant - (p_1^N - 1) \frac{k}{\sigma(k)} + \frac{(p_1^{2N} - 1)}{p_1^{N-1}(p_1 + 1)} \\ &\geqslant \frac{k}{\sigma(k)} + \frac{1}{(p_1 + 1)} \left(1 - \frac{1}{p_1^{N-1}} + \frac{1}{(\sigma(k/p_1) - k/p_1) p_1^{N+1} - 1} \right) . \end{aligned} \tag{0.12}$$

We shall need the expression

$$h_k(x) = \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right) + o(1) , \tag{0.13}$$

where the multiplicative arithmetical function γ_k is defined by

$$\gamma_k(p^m) = \begin{cases} 1 - p & \text{if } p \mid k, \\ 0 & \text{otherwise} \end{cases}$$

(see [HM, Theorem 1 and Lemma 1]), the fact that [HM, (4.1)], if we set

$$H(k,x) := \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{ \frac{x}{n} \right\} \right)$$

then

(0.14) $$H(k,x) = H(k,[x]) - \frac{k}{\sigma(k)} \{x\} + o(1) ,$$

and

LEMMA 0. *We have*

(0.15) $$S_k = \sup_{n \in \mathbf{Z}} H(k,n).$$

Proof. In view of (0.13), (0.14), and the definition of $H(k,x)$, it is sufficient to show that

(0.16) $$\limsup_{N \to \infty, N \in \mathbf{N}} H(k,N) = \sup_{n \in \mathbf{Z}} H(k,n) .$$

When $k = 1$ this is easily verified; when $k \geqslant 2$ and $N \in \mathbf{Z}$ we define for each positive integer i the positive integer $N_i := (|N| + 1)k^i + N$ and we see, since

(0.17) $$\sum_{m \nmid k^i} \frac{\gamma_k(m)}{m} \to 0 \ (i \to \infty) ,$$

and since for every divisor m of k^i we have $\{N_i/m\} = \{N/m\}$, that

(0.18) $$\lim_{i \to \infty} H(k,N_i) = H(k,N) . \quad \square$$

1. PROOF OF THEOREM 1

We first set some terminology. Let $g\colon [1, \infty] \to \mathbf{R}$ be a measurable function, and consider as in [P1]

(1.1) $$D_0(u) = D_{0,g}(u) := \lim_{x \to \infty} \frac{1}{x} \mu \{t \in [0,x], g(t) \leqslant x\} ,$$

and

(1.2) $$D_0(u^+) := \lim_{\substack{v \to u \\ v \in E}} D_0(v) , \quad D_0(u^-) := \lim_{\substack{v \to u \\ v \in E}} D_0(v) ,$$

where μ denotes the Lebesgue measure and E the set of values for which D_0 exists. In case D_0 exists almost everywhere we say, following A. Wintner [W, p. 537], that *g possesses an asymptotic distribution function.* If (and only if) this is so we define an associated function $D = D_g : \mathbf{R} \to [0, 1]$ by

$$(1.3) \qquad D(u) := \frac{1}{2}\left(D_0(u^+) + D_0(u^-)\right).$$

And it is this function D we call *the asymptotic distribution function of g.* The convention is of course abusive[2]; we point out however that D_0 exists and coincides with D at least wherever D is continuous (which, since D is a distribution function, is the case almost everywhere).

The first two statements of Theorem 1, $D = D_h$ exists and is continuous, are proved through a straightforward application of two theorems from [P1].

Indeed, it is easy to see that

$$(1.4) \qquad \sum_{n \leqslant x} \gamma_k(n) = O((\log x)^{\omega(k)}) = 0 \cdot x + o(x)$$

holds, and that for any function $z = z(x) \to \infty$ $(x \to \infty)$ (and in particular for a slowly increasing function), we have

$$(1.5) \qquad H(k, x) = \sum_{n \leqslant z} \frac{\gamma_k(n)}{n}\left(-\psi\left(\frac{x}{n}\right)\right) + o(1),$$

where $\psi(y)$ denotes the function $\{y\} - \frac{1}{2}$ which satisfies

$$(1.6) \qquad \int_0^1 \psi(t)\,dt = 0 .$$

In the notation of [P1] the properties (1.4) through (1.6) are expressed by writing $h \in C_z(\psi_k, -\psi)$. Thus from Theorem 4 of that paper we have the existence of D_h. And since ψ is odd almost everywhere Theorem 5 of [P1] tells us that D_h is symmetric.

We pass now to the third assertion of the theorem, namely that $I_k = -S_k$. We denote by S the bounded support of D_h and by $-s$ and s its

[2]) Its purpose is to ensure that D be *normalized,* i.e. that the relation

$$D(u) = \frac{1}{2}\left(D(u^+) + D(u^-)\right)$$

hold for every real number u.

greatest lower bound and least upper bound: we have $I_k \leqslant -s < s \leqslant S_k$. We show that

$$I_k = -s = -S_k \tag{1.7}$$

holds by ensuring that

$$0 < D_h(\alpha) < 1 \quad \text{for every} \quad \alpha \in (I_k, S_k)\ . \tag{1.8}$$

We prove here that $D_h(S_k - \varepsilon) < 1$ for every $\varepsilon > 0$; the rest of the proof is similar. There is an increasing sequence of natural numbers n_i with $H(k, n_i) \to S_k$ $(i \to \infty)$, and thus we may select some natural number N satisfying

$$H(k, N) > S_k - \frac{\varepsilon}{4} \tag{1.9}$$

and

$$\frac{1}{2} \sum_{n > N} \frac{|\gamma_k(n)|}{n} < \frac{\varepsilon}{4}\ . \tag{1.10}$$

Hence if we define

$$H^*(k, N, M) := \sum_{n \leqslant N} \frac{\gamma_k(n)}{n} \left(\frac{1}{2} - \left\{\frac{M}{n}\right\}\right)\ . \tag{1.11}$$

we have

$$H^*(k, N, N) > S_k - \frac{\varepsilon}{2}\ . \tag{1.12}$$

Also, if L is the least common multiple of the integers $1, 2, \ldots, N$, then

$$H^*(k, N, mL + N) = H^*(k, N, N) \tag{1.13}$$

for every integer m, and it follows from (1.12) and (1.10) that

$$H(k, mL + N) > S_k - \frac{3\varepsilon}{4} \tag{1.14}$$

for every integer m. Now since $D_{0,h}$ exists and coincides with D_h almost everywhere we can find two numbers β and γ satisfying

$$S_k - \varepsilon \leqslant \beta < \beta + \frac{\varepsilon}{5} \leqslant \gamma \leqslant S_k - \frac{3\varepsilon}{4} \tag{1.15}$$

and

$$D_h(\delta) = D_{0,h}(\delta) \quad (\delta = \beta \text{ or } \gamma) . \tag{1.16}$$

In view of (0.14) this implies that

$$\begin{aligned} 1 - D_h(S_k - \varepsilon) &\geqslant D_h\left(S_k - \frac{3\varepsilon}{4}\right) - D_h(S_k - \varepsilon) \\ &\geqslant D_h(\gamma) - D_h(\beta) = D_{0,h}(\gamma) - D_{0,h}(\beta) \geqslant \frac{1}{L} \cdot \frac{\varepsilon}{5} \cdot \frac{\sigma(k)}{k} . \quad \square \end{aligned} \tag{1.17}$$

Remark. I studied in [P2] an error term associated with the k-th Jordan totient function (for $k \geqslant 2$), that can be expressed in terms of the function

$$g_k(x) := - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \psi\left(\frac{x}{n}\right) , \tag{1.18}$$

where μ denotes the Moebius function, and I proved by a direct method that

$$\liminf_{x \to \infty} g_k(x) = - \limsup_{x \to \infty} g_k(x) . \tag{1.19}$$

This can also be obtained by an argument similar to the above proof.

2. The case $\omega(k) = 2$

In this section we obtain an estimate more general than (0.10) of Theorem 2.

Theorem 2′. *Let* $k = pq$ *where* $p < q$ *and* p *and* q *are prime numbers, and let* $d = q - ps$ *with* $1 \leqslant d \leqslant p - 1$ *be the remainder of the Euclidean division of* q *by* p. *Then we have*

$$S_k \geqslant \frac{k}{\sigma(k)} + \frac{1}{(p+1)} - \frac{pd}{(p+1)(q+1)} + \frac{(p+1)(p-2)(q-1)}{p^2 q} . \tag{2.1}$$

The right side of (2.1) is easily seen to exceed $k/\sigma(k)$ for any p and q. And in the special case where $p = 2$ it reduces to $\left(q - \frac{1}{3}\right)/(q+1)$.

Proof. Let N be a positive integer. We define, modulo $p^N q^N$, the integer $x = x_N$ by the system of congruences

$$\begin{cases} x \equiv -1(p^N) \\ x \equiv -d - 1(q^N) . \end{cases} \tag{2.2}$$

We have, for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant N$,

$$x \equiv s_{i,j} q^j - d - 1 (p^i q^j) \quad \text{where} \quad \begin{cases} s_{1,1} = 1 \\ 1 \leqslant s_{i,j} \leqslant p^i - 1 \,, \end{cases} \tag{2.3}$$

whence

$$\begin{aligned} H(k,x) \geqslant & \frac{1}{2} + \sum_{i=1}^{N} \frac{(1-p)}{p^i} \left(-\frac{1}{2} + \frac{1}{p^i} \right) + \sum_{j=1}^{N} \frac{(1-q)}{q^j} \left(-\frac{1}{2} + \frac{d+1}{q^j} \right) \\ & + \frac{(p-1)(q-1)}{pq} \left(\frac{1}{2} - \frac{q-d-1}{pq} \right) \\ & + \sum_{\substack{1 \leqslant i,j \leqslant N \\ (i,j) \neq (1,1)}} \frac{(p-1)(q-1)}{p^i q^j} \left(\frac{1}{2} - \frac{(p^i - 1) q^j - d - 1}{p^i q^j} \right) + o_N(1) \,. \end{aligned} \tag{2.4}$$

The right side of (2.4) tends to the right side of (2.1) as $N \to \infty$, and the theorem is proved in virtue of (0.15). □

PROOF OF THEOREM 3

The function f_r defined in (0.11) satisfies, provided $r \geqslant 3$,

$$f_r(p_2, \ldots, p_r) < f_{r-1}(p_2, \ldots, p_{r-1}) \leqslant p_2 \,, \tag{3.1}$$

and thus the condition

$$f_r(p_2, \ldots, p_r) \geqslant x \tag{3.2}$$

implies, for any x, that

$$p_2 \begin{cases} > x & \text{if} \quad r \geqslant 3 \,, \\ \geqslant x & \text{if} \quad r = 2 \,. \end{cases} \tag{3.3}$$

Also note that, since

$$\sum_{n=1}^{\infty} \frac{\gamma_k(n)}{n} = \prod_{p \mid k} \left(1 + (1-p) \sum_{i \geqslant 1} \frac{1}{p^i} \right) = 0 \,, \tag{3.4}$$

we have in fact

$$H(k,x) = - \sum_{n \geqslant 1} \frac{\gamma_k(n)}{n} \left\{ \frac{x}{n} \right\} \,. \tag{3.5}$$

After these preliminaries let $N \geqslant 1$ be as in (0.11), and define

$$x = x_N := p^N - 1\,, \tag{3.6}$$

where we denote p_1 simply by p. For $r \geqslant 3$ (3.2) and (3.3) imply that

$$p_2 > x\,, \tag{3.7}$$

and (3.7) clearly remains true for $r = 2$ if $p \neq 2$. Hence

$$\begin{aligned} H(k,x) &= (p-1)\left(\frac{p-1}{p^2} + \frac{p^2-1}{p^4} + \ldots + \frac{p^{N-1}-1}{p^{2N-2}}\right) - (p^N - 1 \sum_{n \geqslant N} \frac{\gamma_k(n)}{n^2} \\ &= -\frac{(p^N-1)\,(p^{N-1}-1)}{p^{N-1}(p+1)} - (p^N-1) \sum_{n \geqslant 2} \frac{\gamma_k(n)}{n^2} \\ &= (p^N-1)\left(\frac{(1-p^{N-1})}{(p+1)p^{N-1}} - \prod_{p \mid k}\left(1 + (1-p)\sum_{i \geqslant 1} \frac{1}{p^{2i}}\right) + 1\right) \\ &= (p^N-1)\left(\frac{p^N+1}{(p+1)p^{N-1}} - \frac{k}{\sigma(k)}\right). \end{aligned} \tag{3.8}$$

Now when a rational number P/Q is less than an integer M, we may conclude that $M - P/Q \geqslant 1/Q$. Thus from (0.11) we have

$$\frac{p}{p+1} \cdot \frac{\sigma(k)}{k} = \frac{\sigma(k/p)}{k/p} \geqslant \frac{p^{N+1} - (\sigma(k/p) - k/p)^{-1}}{p^{N+1} - 1 - (\sigma(k/p) - k/p)^{-1}}\,, \tag{3.9}$$

whence from (3.8)

$$H(k,x) \geqslant \frac{k}{\sigma(k)} + \frac{1}{(p+1)}\left(1 + \frac{1}{(\sigma(k/p) - k/p)p^{N+1} - 1} - \frac{1}{p^{N-1}}\right). \tag{3.10}$$

On appealing to Lemma 0 this concludes the proof of the theorem. □

Last Remark. Neither of the estimates (2.1) (of Theorem 2′) and (0.12) (of Theorem 3) is better than the other in all cases considered by both theorems. For instance in the case where $k = pq = p(p + d)$ with p and q odd primes and $2 \leqslant d \leqslant p - 2$, there is some positive number ε depending on p, satisfying

$$\frac{13/4}{\sqrt{p}} < \varepsilon < \frac{8.06}{\sqrt{p}}\,, \tag{3.11}$$

and such that (2.1) is better than (the first estimate of) (0.12) if $d < 2\sqrt{p} + 2 + \varepsilon$, and is not as good if $d > 2\sqrt{p} + 2 + \varepsilon$.

ADDED IN PROOF. Recently, S. D. Adhikari and K. Soundararajan gave a much simpler proof of (0.9) than mine in "Towards the exact nature of a certain error term, II" (preprint).

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Yves-F. S. Pétermann

Section de Mathématiques
Université de Genève
C.P. 240
CH-1211 Genève 24 (Suisse)