0. Introduction

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YANGIANS AND R-MATRICES

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0. INTRODUCTION

Quantum groups arose as the natural language in which to formulate certain techniques which had been developed to construct and solve integrable quantum systems (see [6]). The most important examples are quantum Kac-Moody algebras and Yangians. Representations of both types of quantum groups are closely related to solutions of the quantum Yang-Baxter equation (QYBE). In particular, finite-dimensional irreducible representations of Yangians give rational solutions of the QYBE. In this paper we shall give a complete and elementary description of all the irreducible finite-dimensional representations of the Yangian associated to $\mathfrak{sl}_2$. The importance of this example is analogous to that of $\mathfrak{sl}_2$ itself, whose representation theory is the foundation for that of an arbitrary semi-simple Lie algebra.

A quantum group is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ in the category of (not necessarily co-commutative) Hopf algebras. More precisely, let $\mathbf{C}[[h]]$ be the algebra of power series in an indeterminate h; we shall have occasion to use the grading on $\mathbf{C}[[h]]$ obtained by setting $\deg h = 1$. Then, a quantum group is a Hopf algebra A over $\mathbf{C}[[h]]$ such that one has an isomorphism of Hopf algebras

$$A/hA \cong U(\mathfrak{g}) \,. \tag{0.1}$$

Further, A is required to be complete and topologically free as a $\mathbf{C}[[h]]$-module (the latter condition means that A/h^nA is a free $\mathbf{C}[[h]]/(h^n)$-module for all $n \geqslant 1$). We shall sometimes refer to A as a quantization of $U(\mathfrak{g})$. One thinks of A as a "quantum" object and interprets the isomorphism (0.1) as meaning that $U(\mathfrak{g})$ is obtained from A by taking the "classical limit $h \to 0$".

Let $\Delta: A \to A \otimes A$ be the co-multiplication map of A, $\sigma: A \otimes A \to A \otimes A$ the switch of the two factors, and set $\Delta' = \sigma\Delta$. For any $x \in U(\mathfrak{g})$, choose $a \in A$ such that $a \equiv x (\operatorname{mod} h)$. Then

$$\frac{\Delta(a) - \Delta'(a)}{h} \pmod{h}$$

is a well-defined element $\delta(x) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The map $\delta: U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is determined by its restriction to $\mathfrak{g}$, which maps $\mathfrak{g}$ into $\mathfrak{g} \otimes \mathfrak{g}$, by the formula

$$\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b) .$$

Moreover, δ is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, and the dual map $\delta^*: \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ gives $\mathfrak{g}^*$ the structure of a Lie algebra. An important special case is that in which δ is a 1-coboundary, which means that, for some $r \in \mathfrak{g} \otimes \mathfrak{g}$ we have

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r]$$

for all $x \in \mathfrak{g}$. The dual of this map δ defines a Lie bracket on $\mathfrak{g}^*$ if and only if

$$r^{12} + r^{21} \in \mathfrak{g} \otimes \mathfrak{g} \tag{0.2}$$

and

$$< r, r > \equiv [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \tag{0.3}$$

are $\mathfrak{g}$-invariant. Here, $r^{12} = r \otimes 1 \in U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})$ etc. In particular, equations (0.2) and (0.3) are satisfied if r is skew and satisfies the classical Yang-Baxter equation (CYBE)

$$< r, r > = 0 .$$

Yangians arise from the case $\mathfrak{g} = \mathfrak{a}[t]$, where $\mathfrak{a}$ is a finite-dimensional complex simple Lie algebra and t an indeterminate. Note that setting $\deg t = 1$ makes $U(\mathfrak{g})$ a graded Hopf algebra. Then, r becomes a function of two variables t_1, t_2 with coefficients in $\mathfrak{a} \otimes \mathfrak{a}$; the skewness condition is now

$$r^{12}(t_1 - t_2) + r^{21}(t_2 - t_1) = 0 , \tag{0.4}$$

and the CYBE becomes

$$\begin{aligned} [r^{12}(t_1 - t_2), r^{13}(t_1 - t_3)] &+ [r^{12}(t_1 - t_2), r^{23}(t_2 - t_3)] \\ &+ [r^{13}(t_1 - t_3), r^{23}(t_2 - t_3)] = 0 . \end{aligned} \tag{0.5}$$

The simplest solution of equations (0.4) and (0.5) is

$$r(t_1, t_2) = \frac{\Omega}{(t_1 - t_2)} , \tag{0.6}$$

where

$$\Omega = \sum_{\lambda} (I_\lambda \otimes I_\lambda) , \tag{0.7}$$

and $\{I_\lambda\}$ is an orthonormal basis of $\mathfrak{a}$ with respect to a fixed $\mathfrak{a}$-invariant bilinear form on $\mathfrak{a}$. According to Drinfel'd [4], there is, up to isomorphism, a unique quantization A of $U(\mathfrak{a}[t])$ such that:

(1) A is a graded $\mathbf{C}[[h]]$-algebra and (0.1) is an isomorphism of graded algebras;

(2) $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$ for $x \in \mathfrak{a}[t]$, where r is given by (0.6).

The algebra A is generated by elements x, $J(x)$ for $x \in \mathfrak{g}$, whose "classical limits" are the generators x, xt of $\mathfrak{a}[t]$ (see Definition 1.1 for a precise description of A). The defining relations of A only involve polynomials in h, and hence it makes sense to specialize to a particular value of h. The resulting Hopf algebra A_h over $\mathbf{C}$ is easily seen to be independent of h, up to isomorphism, as long as $h \neq 0$; setting $h = 1$ gives the Yangian $Y(\mathfrak{a})$. (More precisely, $Y(\mathfrak{a}) = A'/(h-1)A'$, where A' is the algebraic direct sum of the homogeneous components of A.)

For the Lie algebra $\mathfrak{a}[t]$, one has the evaluation homomorphisms $\varepsilon_a : \mathfrak{a}[t] \to \mathfrak{a}$ for any $a \in \mathbf{C}$. Pulling back a representation of $\mathfrak{a}$ by such a map gives a so-called "evaluation representation" of $\mathfrak{a}[t]$, and it is known [1], [2] that every finite-dimensional irreducible representation of $\mathfrak{a}[t]$ is isomorphic to a tensor product of evaluation representations. When $\mathfrak{a} = \mathfrak{sl}_2$, the evaluation homomorphisms admit "quantizations" $\varepsilon_a : Y(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)$ such that

$$\varepsilon_a(x) = x\,, \qquad \varepsilon_a(J(x)) = ax\,.$$

Evaluation representations of $Y(\mathfrak{sl}_2)$ can now be defined just as for $\mathfrak{sl}_2[t]$. One of the main results of this paper is:

THEOREM. *Every finite-dimensional irreducible representation of $Y(\mathfrak{sl}_2)$ is isomorphic to a tensor product of evaluation representations.*

The representation theory of $Y(\mathfrak{a})$ is closely related to the quantum Yang-Baxter equation (QYBE):

$$\begin{aligned} &R^{12}(t_1 - t_2)R^{13}(t_1 - t_3)R^{23}(t_2 - t_3) \\ &= R^{23}(t_2 - t_3)R^{13}(t_1 - t_3)R^{12}(t_1 - t_2)\,. \end{aligned} \tag{0.8}$$

Here, the function $R(t)$ is usually understood to take values in $\mathrm{End}(V \otimes V)$ for some finite-dimensional vector space V, although it makes sense when R takes values in $B \otimes B$ for any associative algebra B. In view of equation (0.6), it is natural to look for solutions of the form

$$R(t) = 1 + t^{-1}(\rho(I_\lambda) \otimes \rho(I_\lambda)) + \sum_{k=2}^{\infty} R_k t^{-k} \tag{0.9}$$

for some representation ρ of $\mathfrak{sl}_2$ on V and some $R_k \in \mathrm{End}(V \otimes V)$. If $\tilde{\rho}$ is an extension of ρ to $Y(\mathfrak{a})$ and $\mathfrak{R} \in Y(\mathfrak{a}) \otimes Y(\mathfrak{a})$ satisfies equation (0.8), then $R = (\tilde{\rho} \otimes \tilde{\rho})(\mathfrak{R})$ will be a solution of the QYBE with values in $\mathrm{End}(V \otimes V)$. Drinfel'd proved [4] that there exists an essentially unique "universal R-matrix" $\mathfrak{R}$, that the resulting matrix-valued solutions $R(t)$ are rational, and that every rational solution of the QYBE of the form (0.9) arises in this way.

Unfortunately, the universal R-matrix for $Y(\mathfrak{a})$ is not known explicitly, so in section 5 we shall give an alternative construction of rational solutions of the QYBE, which relates them to intertwining operators between tensor products of certain representations of $Y(\mathfrak{a})$. Although this is presumably well-known, it does not seem to have appeared in print before. We use this technique to write down explicitly the solutions of the QYBE associated to all the finite-dimensional irreducible representations of $Y(\mathfrak{sl}_2)$. These solutions were first written down by Kulish, Reshetikhin and Sklyanin [9], but without proof (according to these authors "the proof... is lengthy"). We obtain the R-matrices with minimal computation, and the fact they satisfy the QYBE is a consequence of general results.

The relation between R-matrices and Yangians can be inverted. Let $R(t)$ be a rational solution of the QYBE arising from an irreducible representation of $Y(\mathfrak{a})$ on $\mathbf{C}^n$. To this one associates a Hopf algebra Y_R generated by elements $\{t_{ij}^{(k)}\}$, $1 \leqslant i, j \leqslant n$, $k = 1, 2, \ldots$ Let $T(s)$ be the matrix

$$T(s)_{ij} = \delta_{ij} + \sum_k t_{ij}^{(k)} s^{-k} .$$

Then, the relations are

$$(T(t) \otimes id)\ (id \otimes T(s)) R(t-s) = R(t-s)\ (id \otimes T(s))\ (T(t) \otimes id) ,$$

and the co-multiplication map is given by

$$\Delta(T(s)_{ij}) = \sum_k T(s)_{ik} \otimes T(s)_{kj} .$$

Then, $Y(\mathfrak{a})$ is a quotient of Y_R by an ideal generated by certain group-like elements of the centre of Y_R (called "quantum determinants"). This approach to Yangians appears implicitly in the early work on quantum inverse scattering theory (see [6] for an excellent survey) and also explicitly in more recent work (see, for example, Kirillov and Reshetikhin [7]).

We conclude this introduction with some remarks on the literature. There is a classification of the finite-dimensional irreducible representations of Yangians analogous to that for complex semisimple Lie algebras; this will be described in section 2 and used to obtain our main theorem, which provides a concrete model for the representations. The classification was first obtained

by Tarasov [12], [13], for the case of $Y(\mathfrak{sl}_2)$, using ideas of Korepin [8], and was extended by Drinfel'd [5] to the general case. The evaluation representations and their tensor products appeared implicitly in the work of Kulish, Reshetikhin and Sklyanin [9] mentioned above, but they did not prove that all finite-dimensional irreducible representations are of this form. Our determination of the precise conditions under which such tensor products are irreducible is also new.

One of the difficulties of this subject is the unfamiliarity of the language in which many of the fundamental papers are written, which is that of quantum inverse scattering theory and exactly solvable models in statistical mechanics. We have tried in our presentation to express the results in more conventional mathematical language. In fact, all that is required is some familiarity with the basic techniques of Lie theory.

1. Yangians

We begin with the definition of the Yangian taken from [4]. Let $\{I_\lambda\}$ be an orthonormal basis of $\mathfrak{sl}_2$ with respect to some invariant inner product $(\ ,\ )$; for example, using the trace form

$$(x, y) = \operatorname{trace}(xy) ,$$

one can take the basis $\left\{\dfrac{x^+ + x^-}{\sqrt{2}}, \dfrac{i(x^+ - x^-)}{\sqrt{2}}, \dfrac{h}{\sqrt{2}}\right\}$, where $\{x^+, x^-, h\}$ is the usual basis:

$$[h, x^\pm] = \pm 2x^\pm \quad , \qquad [x^+, x^-] = h .$$

Definition 1.1. The *Yangian* $Y = Y(\mathfrak{sl}_2)$ associated to $\mathfrak{sl}_2$ is the Hopf algebra over $\mathbf{C}$ generated (as an associative algebra) by $\mathfrak{sl}_2$ and elements $J(x)$ for $x \in \mathfrak{sl}_2$ with relations

(1) $[x, J(y)] = J([x, y]), \qquad J(ax + by) = aJ(x) + bJ(y), \ a, b \in \mathbf{C}$,

(2) $$[J(x), J([y, z])] + \text{cyclic permutations of } x, y, z = ([x, I_\lambda], [[y, I_\mu], [z, I_\nu]])\{I_\lambda, I_\mu, I_\nu\} ,$$

(3) $$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] = (([x, I_\lambda], [[y, I_\mu], [[z, w], I_\nu]]) + ([z, I_\lambda], [[w, I_\mu], [[x, y], I_\nu]]))\{I_\lambda, I_\mu, I_\nu\},$$

where repeated indices are summed over and

$$\{x_1, x_2, x_3\} = \sum_\pi x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} ,$$