STATE MODELS FOR LINK POLYNOMIALS

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STATE MODELS FOR LINK POLYNOMIALS

by Louis H. KAUFFMAN

I. INTRODUCTION

This paper gives state models for the oriented (Homfly) ([32], [39], [68]) and semi-oriented (Kauffman) ([48], [49], [43]) polynomials.

The models are state models in the most general spirit of that term. That is, they are summations over diagrammatic states. Each state is obtained from the given link diagram by processes of splicing and labelling. Each state contributes to the summation a product of vertex weights together with a global evaluation. Such a description is parallel to the form of a vertex model or partition function in statistical mechanics [10].

In fact, a number of already known states models are tightly interrelated with statistical mechanics. The author's bracket model for the original Jones polynomial is a knot diagrammatic version of the Potts model (via an ice-model translation) ([44], [51]). The author's states model for the Conway polynomial [42] can be seen as the low temperature limit of a generalized Potts model [57]. Jones and Turaev ([40], [93]) have given models of specializations of both the Homfly and Kauffman polynomials that are also in the form of partition functions (called here Yang-Baxter models, see section 7 of this paper. See also [25], [58]).

One might hope that a state model would make the verification of the existence and properties of a polynomial easy — via the directness of the definition. This is certainly the case for the bracket. And it is also true for the FKT [42] model for the Conway polynomial (modulo a combinatorial theorem, and some algebra). In the case of the Yang-Baxter models, the work is concentrated in seeing that the associated Yang-Baxter "tensor" satisfies certain properties (Yang-Baxter equation and an inversion relation). This part can be clarified using the geometry of link diagrams ([55], [58]).

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The models given in this paper do not share this ease of definition. Their well-definedness is just as easy or difficult as the standard inductive proof of the existence of the polynomial in question. To discriminate them from the Yang-Baxter and other models, I shall call them *skein models.* The skein models are, in fact, seen as translations (into the language of state models) of the recursive process of calculation due to John H. Conway ([16]). The term *skein* is due to Conway. It refers to all the knots and links associated with a given link that are obtained via splicing or switching some of its crossings. Skein calculation uses the knots and links in this skein.

The first skein model was discovered by François Jaeger [33], via a matrix inversion technique. In this paper I show that Jaeger's idea fits into the more general scheme of skein calculation, and hence applies fully to both of the known two-variable skein polynomials, and to their graph embedding generalizations ([56], [74]).

As explained herein, the skein models appear as tautologous — particularly to anyone who has written a computer program to calculate knot polynomials. This is the virtue of our approach! What is significant is that we have, in fact, been in possession of general states models for these polynomials for years. It took Jaeger's observation about standardized basepoints (called here a *template* — see section 2) to show what we already knew.

It is useful to know that general states models exist. And it is very fascinating to compare the form of the general models to the forms of specific models previously known (bracket, FKT, Yang-Baxter). See sections 7, 8, 9 of this paper, particularly section 9 for a discussion of possible physical interpretations. The appendix is a discussion of state model formalism.

II. Skein polynomials

This paper concentrates on two polynomials — the *Homfly* ([24]) or *oriented skein polynomial* and the *Kauffman* or *semi-oriented skein polynomial* ([43], [49]). In both cases it is convenient to first define a polynomial that is an invariant of regular isotopy ([49]), and then normalize the regular isotopy invariant to obtain the corresponding skein polynomial.

NOTATION. *Regular isotopy* is the equivalence relation generated by Reidemeister moves II and III (see Figure 1). Regular isotopy of links K and L is denoted $K \approx L$. *Ambient isotopy* is the equivalence relation generated by all three Reidemeister moves, and will be denoted by $K \sim L$. (See [80] or [14] for a proof that this diagrammatic version of ambient isotopy agrees with the usual definition for embeddings of links in three dimensional Euclidean space.)

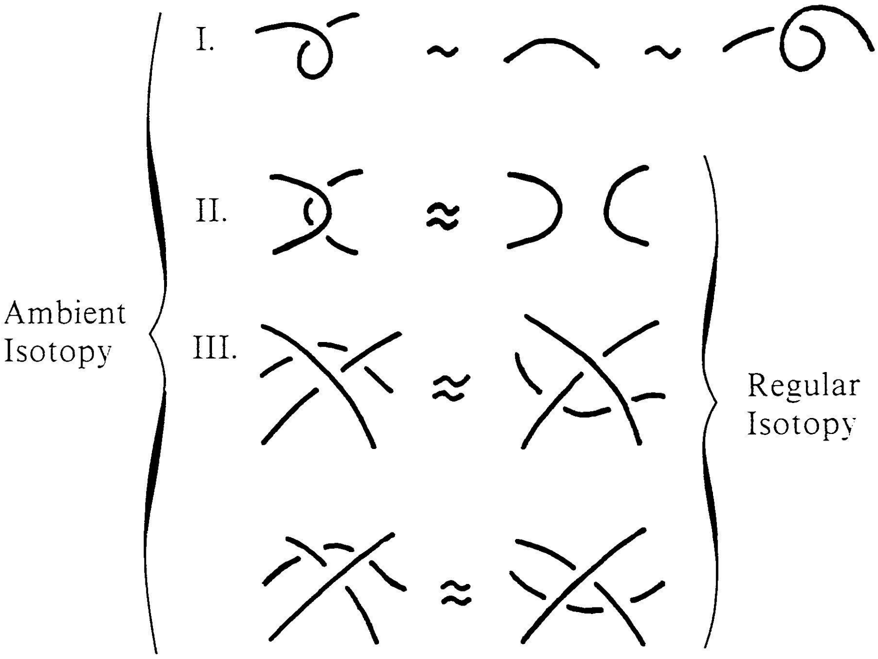


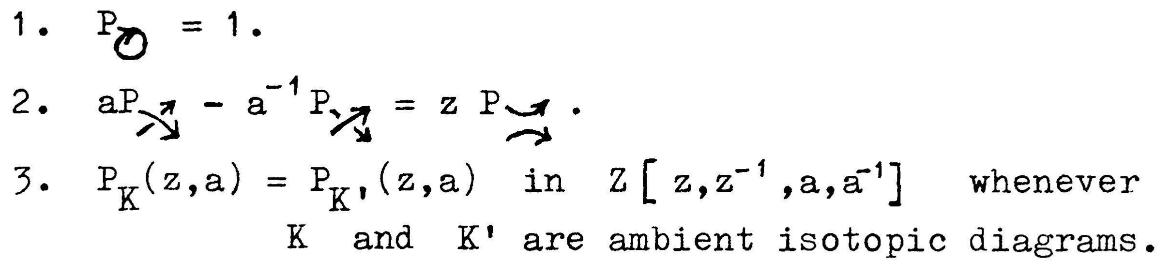
FIGURE 1
The Reidemeister Moves

In an oriented link, each crossing has a *sign* $\varepsilon = +1$ or -1, according to the convention shown in Figure 2.

FIGURE 2
Crossing Signs

The *writhe*, $w(K)$, of an oriented diagram K is the sum of the signs of the crossings in the diagram. The writhe is an invariant of regular isotopy.

The *oriented skein polynomial*, $P_K(z, a)$, is described by two axioms as an ambient isotopy invariant:

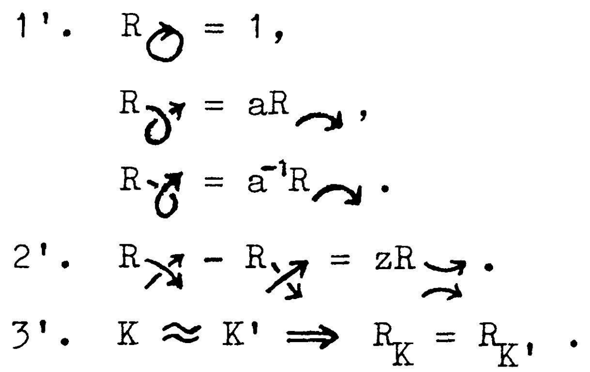


The three small diagrams in axiom 2 stand for parts of larger diagrams that differ only as shown in the small diagrams.

I shall use the following formulation for P:

$$P_K(z, a) = a^{-w(K)} R_K(z, a)$$

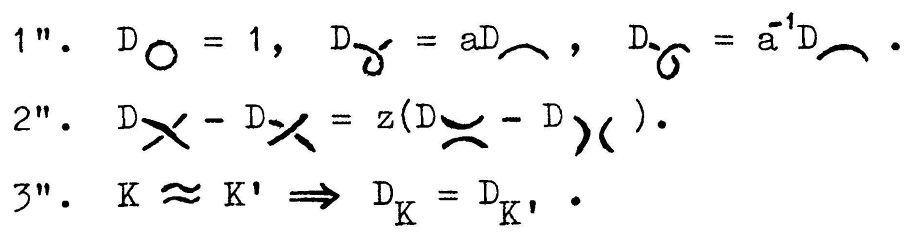
where $w(K)$ is the writhe of K, and $R_K(z, a)$ is a regular isotopy invariant satisfying the axioms:



The identity of axiom 2′. will be referred to as the *Conway identity*. (If $a = 1$ then $R_K(z, 1) = \nabla_K(z)$ is the Conway-Alexander polynomial [16].)

It is known that these axiom systems are consistent, and that they can be used to recursively calculate the polynomials.

I shall also discuss the "Dubrovnik" version of the Kauffman polynomial [49]. This regular isotopy invariant, denoted $D_K(z, a)$, is defined for unoriented links K via the axioms:



The key is to see that an unoriented crossing can distinguish two pairs of local regions (designated A and B) as shown in Figure 3.

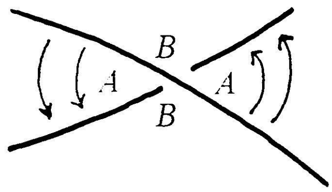


FIGURE 3

The A regions are swept out first as the overcrossing line is turned counterclockwise.

With this in mind, axiom 2. makes sense by associating with each crossing on the left-hand side of the equation the corresponding splice on the right-hand side that joins the A-regions.

The ambient isotopy invariant, Z_K, for the Kauffman (Dubrovnik version) polynomial is given by the formula

$$Z_K(z, a) = a^{-w(K)} D_K(z, a) .$$

where K is an oriented link (D forgets the orientation).

These polynomials can be calculated systematically and recursively by using the concept of the *standard unlink* ([8]):

Given a universe (locally 4-valent plane graph), choose an edge and a direction on that edge. Traverse the universe, *crossing over* at the first passage through each crossing. (Upon returning to a crossing, cross under — so leaving the choice at first passage unchanged.) If this crossing circuit does not use up every crossing in the universe, then choose an edge in the complement of circuits traversed so far, and repeat the process.

The final result is an unlink (each component is unknotted and any two components are unlinked). See Figure 4.

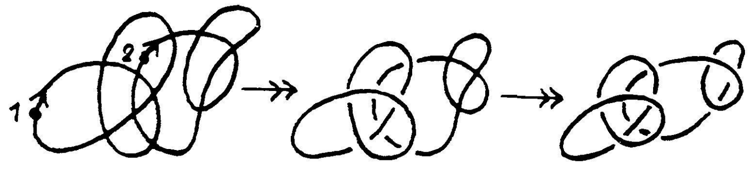


FIGURE 4

The exchange identity (for either polynomial) expresses the value for a given link in terms of the value for a link obtained by switching one crossing, and the value of the polynomial on a link (or links) with fewer crossings. If the switched link is "more unlinked" than the given link, then this procedure proceeds inductively to calculate the polynomial.

The value of the standard unlink is that it can be used as a basis of comparison for the switching process. Having chosen a standard unlink, K, with the same underlying universe as K, we can decide to switch a crossing in K exactly when it *differs* from the corresponding crossing on the standard unlink. Successive switchings then lead toward the standard unlink.

Of course, the calculation process also involves links obtained by splicing crossings of K. For these it is necessary to also choose standard unlinks.

Each standard unlink for K is determined by a choice of base-point and direction of travel along K. Choosing a basepoint is equivalent to choosing an edge from the universe U for K (an edge extends from one crossing to another). We can choose all the basepoints for the whole calculation at once by making a *template*:

Definition 2.1. Let K be a link diagram with underlying universe U. Suppose that U has n edges. A *template for* K is a labelling of the edges of U from the set $\{1, 2, ..., n\}$ when K and U are oriented. If K is unoriented, then a template consists in a labelling of the edges of U, together with an orientation of each edge of U. (Note that there are no compatibility restrictions on these orientations, as in the orienting of a link diagram where travel in the direction of the orientation carries one through a crossing.)

A diagram obtained from K (above) by splicing some crossings can be regarded as having extra 2-valent vertices at the sites of the splice. In this way *the template for* K *is regarded as a template for all diagrams obtained from* K *by either switching or splicing crossings.*

Let K be a given link diagram and T a template for K. The link components of K correspond to circuits in the underlying universe U. (Such circuits cross at a crossing, and are characterized by this property.) Let L be a link component of K.

Definition 2.2. The *index of* L *with respect to the template* T, $I(L, T)$, is the least index in T on the circuit corresponding to L. Let $\mathrm{Ind}(K)$ denote the collection of indices of components of K, arranged in increasing order.

With this notion of index we can specify the algorithm to compute either polynomial:

Template Skein Algorithm

0. This algorithm produces a tree rooted at the given link diagram K, with a collection of unlinks at its ends (farthest branches). The nodes of this tree are link diagrams.

1. Begin by letting i denote the least index in Ind(K).

2. Using the basepoint specified by the index i, find the first crossing that is an undercrossing (in the direction of travel from the basepoint). Call this crossing the *active crossing.*

3. Produce the switched and spliced diagrams associated to the active crossing (two diagrams in the oriented case, three in the semi-oriented case). *Regard the diagrams so produced as nodes on the tree, connected by edges of the tree with the source diagram* K.

4. If all crossings are inactive from a given basepoint, choose the next basepoint, j, in the order specified by Ind(K). Using the basepoint specified by the index j, find the first crossing *of components of index at least equal to* j that is an undercrossing (in the direction of travel from the basepoint). Produce new nodes (see 3. above) from this crossing.

5. If all crossings are inactive for all basepoints, then K is a standard unlink. Call this an *end-node* in the tree.

6. Apply this process to each produced node until the tree is complete.

The value of the polynomial on any unlink is given by the formulas:

$$R_K = a^{w(K)}\delta^{|K|-1}$$

$$D_K = a^{w(K)}\mu^{|K|-1}$$

$$(K \text{ unlinked}, \ |K| = \text{number of components of } K)$$

$$\delta = (a - a^{-1})/z$$

$$\mu = 1 + (a - a^{-1})/z\,.$$

(This follows directly from the axioms.)

Call this tree the *template skein for* K *relative to* T. See Figure 5.

Definition 2.3. Let U be a universe with template T. A link L is said to be a *standard unlink relative to* T if the tree produced by the template skein algorithm for L relative to T consists in a single node.

Letting $E_T(K)$ denote the set of end-nodes (standard unlinks) produced by the template skein algorithm for K relative to T, we have that

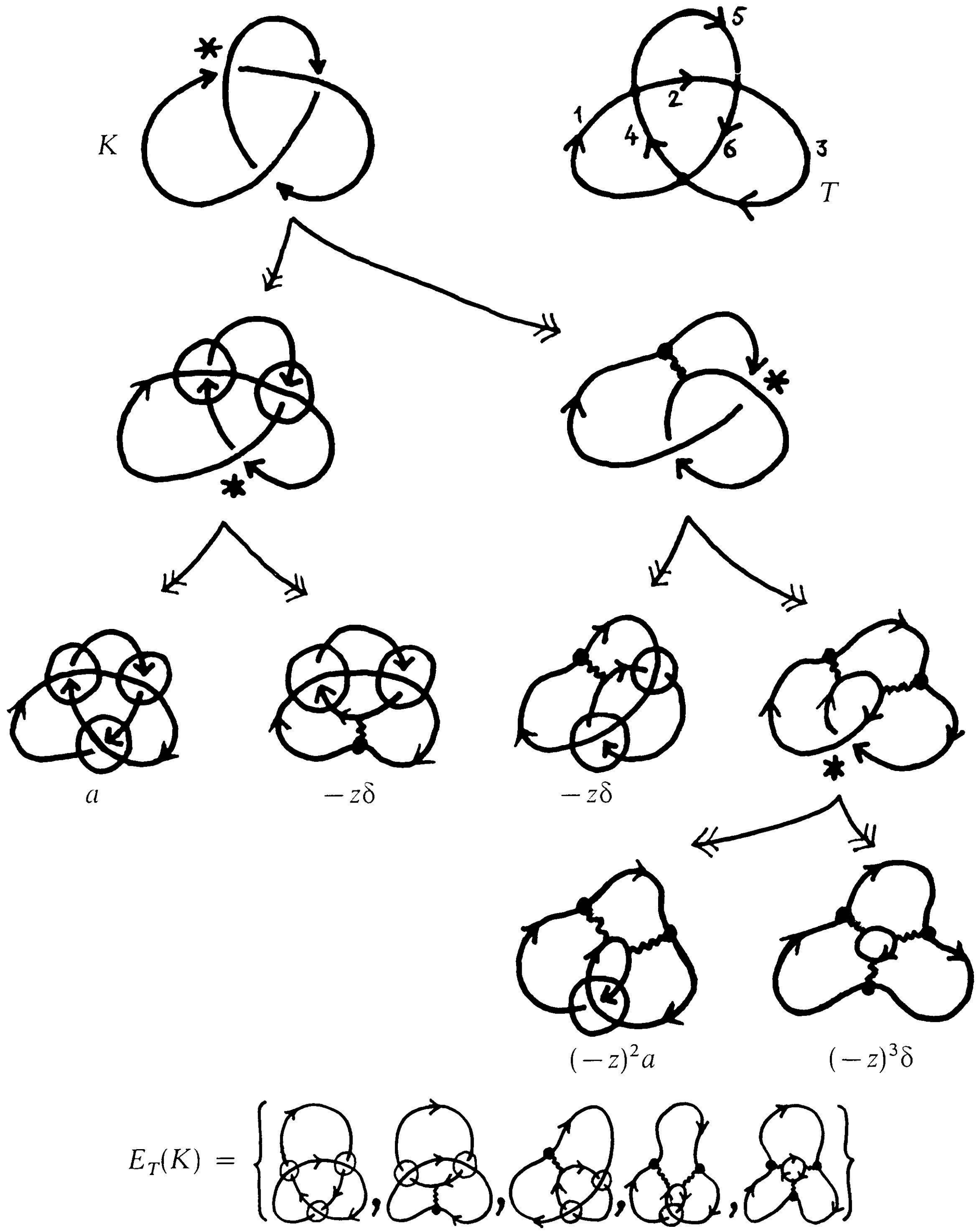
$$R_K = \sum (-1)^{t^-(L)} z^{t(L)} a^{w(L)} \delta^{|L|-1}$$

summation over $L \in E_T(K)$ (E for oriented case, E for unoriented case) and

$$D_K = \sum (-1)^{t^-(L)} z^{t(L)} a^{w(L)} \mu^{|L|-1}$$

summation over $L \in E_T(K)$, where $t(L)$ denotes the number of splices performed to obtain L from K, and $t_-(K)$ is the number of splices from a crossing of negative sign in the Homfly case. In the Dubrovnik case, $t_-(K)$ denotes the number of splices that are of "B-type" (see Figure 3). This formula follows directly from the exchange identity, and the form of calculation.

The next two sections will characterize and reformulate the elements of $E(K)$, and thus construct the skein models.



$$R_K = a - z\delta - z\delta + z^2a - z^3\delta = a^{-1}z^2 + (za^{-1} - a)$$

FIGURE 5

Template Skein Algorithm

III. A SKEIN MODEL FOR THE HOMFLY POLYNOMIAL

In this section K is an *oriented* diagram with underlying universe U. A template T is given for U.

Consider the template skein algorithm of section 2. At each stage in the algorithm an active crossing produces (oriented case) a spliced diagram and a diagram with a switched crossing. It is easy to see that *once a crossing has been switched by the template skein algorithm it remains unchanged in all diagrams in the tree that are descendants of the switched diagram.* (The crossing becomes inactive for all the descendants.) Consequently, we may encircle such a crossing to indicate its inactivity, as was done in Figure 5.

By the same token, consider a crossing that has been spliced from an active crossing by the template skein algorithm. As shown in Figures 5 and 6, if the splice occurs from a positive crossing, then in the splice — *the first passage from basepoint goes through the right hand (or lower) strand of the splice.* If the splice occurs from a negative crossing, then in the splice — *the first passage from basepoint goes through the left hand (or upper) strand of the splice.* Here right and left refer to an observer who stands between the strands of the splice, facing in the direction of the orientation. First passage is indicated in Figures 5 and 6 by a dot on the first passage strand, coupled by a wavy line between the strands of the splice. *This location of the first passage strand is inherited by all the descendants of the given diagram.* This property follows from the precedence structure of the template skein algorithm. (Previously spliced and switched crossings are visited first by the algorithm, before it finds the next activity.)

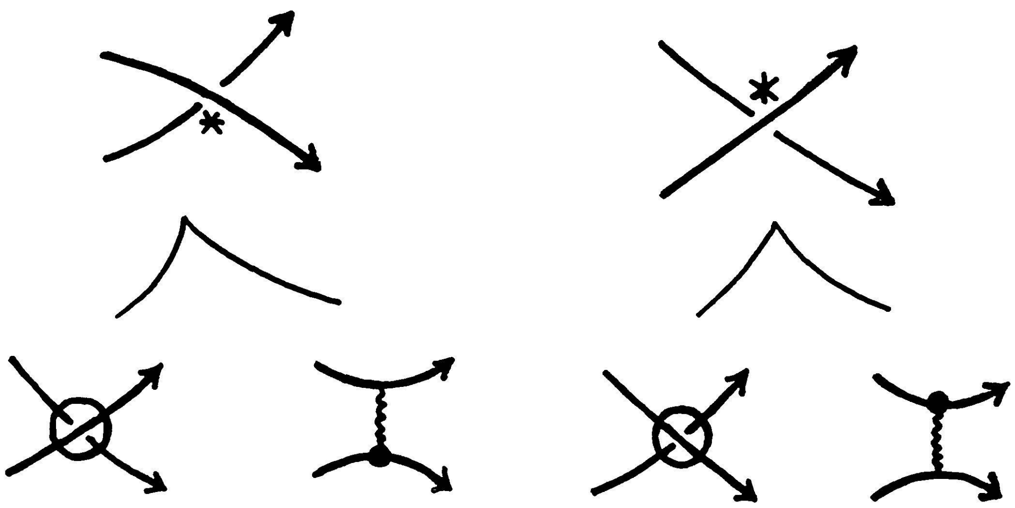


FIGURE 6

As a result of these two remarks we see that *each end node in the tree produced by the template skein algorithm is unique*. That is there are no repetitions among the unlinks produced by the algorithm. Furthermore, each end-node has each of its spliced sites decorated either by a dot and wavy line — indicating first passage on the right or on the left. *This first passage refers to the action of the template skein algorithm applied to the given end-node*. In each case, first passage on the right corresponds to a positive crossing in the original link K, while first passage on the left corresponds to a negative crossing in K.

Definition 3.1. Let K be a given link diagram. Let U be the universe for K, and T a template for U. Let L be a standard unlink relative to T (see Definition 2.3). L is said to be *admissible with respect to* K if at each splice in L, the first passage of the template skein algorithm is on the right hand strand for positive crossings of K and on the left hand strand for negative crossings of K.

The preceding remarks may be summarized by the proposition.

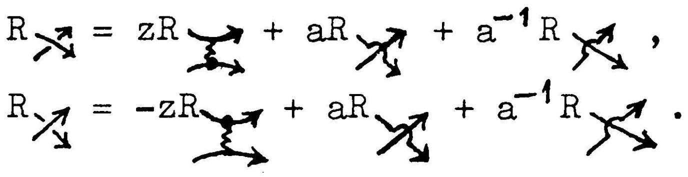
PROPOSITION 3.2. *Let* K *be a link diagram with universe* U, *and a given template* T *for* U. *Then the admissible unlinks relative to* K *are in one-to-one correspondence with the end-nodes of the template skein algorithm applied to* K *(using the template* T*).*

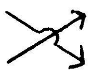
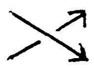
We can now write the state summation

$$R_K = \sum (-1)^{t_-(L)} z^{t(L)} a^{w(L)} \delta^{|L|-1},$$

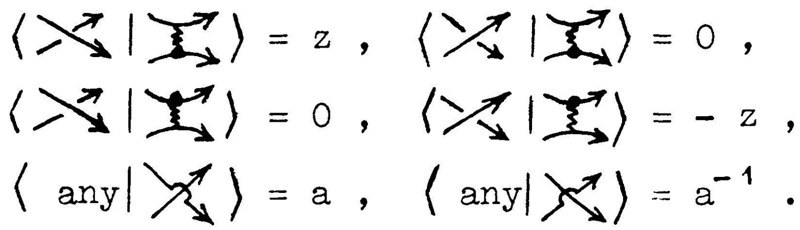
summation over $L \in A(K, T)$, where $A(K, T)$ denotes the set of admissible unlinks relative to T. As in section 2, $t_-(L)$ denotes the number of splices of negative crossings in K, $t(L)$ denotes the number of splices of crossings in K needed to obtain L. $w(L)$ denotes the writhe of L, $|L|$ denotes the number of components of L, and $\delta = (a - a^{-1})/z$.

This state summation can be expressed symbolically by the equations:



Here the symbol  means the same as  except that I use this notation to differentiate a state in $A(K, T)$ from the original link K. Expansion via these equations leads to a large collection of state diagrams many of them inadmissible. However, an inadmissible diagram has some of its vertex weights equal to zero:

$$R_K = \sum_{L \in D(K,\, T)} \langle K \mid L \rangle \, \delta^{|L|-1}$$



Here $D(K, T)$ denotes a set of diagrammatic states (obtained by replacing crossings of K by one of the four local glyphs indicated by the expansion equations). Each state in $D(K, T)$ must satisfy:

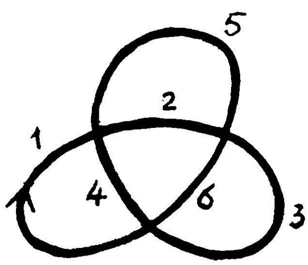
1. Dotted segments must denote first passage relative to the template T.

2. Over and under-crossing indications () must indicate first passage relative to the template T. Notations such as

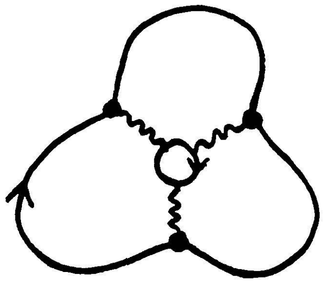
denote the local vertex weights. $\langle K \mid L \rangle$ denotes the product of vertex weights from the sites of the diagrammatic state L.

Note that $D(K, T)$ and $A(K, T)$ are in one-to-one correspondence. Hence this diagrammatic description of the model is equivalent to the summation involving the admissible unlinks.

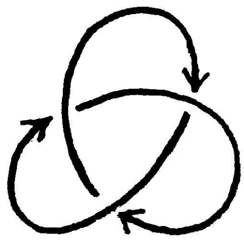
In these expansion formulas the writhe contribution is picked up by the coefficients a and a^{-1}. The decorations and over-passes on an admissible state must be compatible with the template and circuit structure. Thus for the template



the state



is admissible and contributes $(-z)^3\delta$ to the summation for the negative trefoil



IV. A Skein Model for the Kauffman polynomial

The work of section 3 goes over essentially verbatim for the Dubrovnik version of the Kauffman polynomial. Recall that in this context a template is obtained by first orienting the edges of the universe U underlying the unoriented K, and then labelling the edges of U.

The chart in Figure 7 shows the cases of admissible splices at crossings (with respect to the skein template algorithm). Each splice has been labelled with its corresponding vertex weight. Note that a splice is admissible if it indicates the form of passage that is obtained from an approach to the crossing that meets it as an under-crossing. Such approaches give active crossings in the skein template algorithm.

I have retained only the arrow for the first passage after each split, because the orientation on the other edge may change under the direction of the template. *The crossings are oriented because each end-node (unlink) produced by the skein template algorithm acquires an orientation from the directions of travel given by the template.*

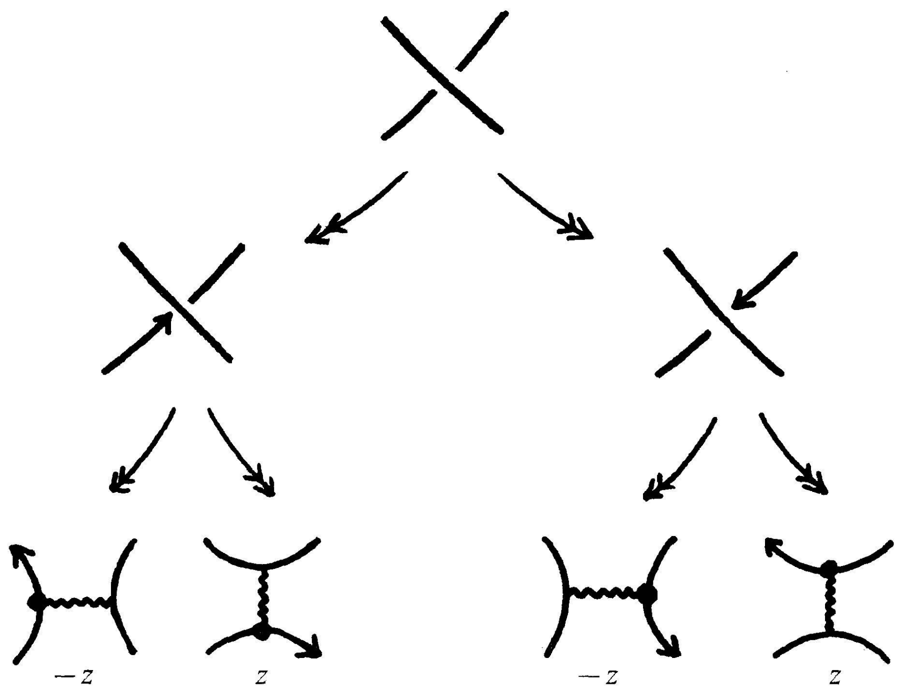
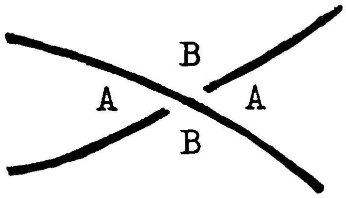


Figure 7

In order to understand the pattern of these admissible splices, consider an unoriented crossing



I have labelled two of its regions A. These are swept out by turning the overcrossing line counterclockwise. The other two regions are labelled B. The *A-splice* of the crossing is that splice that joins the two regions labelled A. The *B-splice* joins the two regions labelled B.

We then see that a passage is admissible in the A-splice if it occurs on the *right* for an observer who stands in between the strands, facing in the direction of the passage from basepoint. Similarly, the admissible B-splices are on the left for such an observer.

Call an admissible splice *negative* if it is of B-type. (This receives a $(-z)$ in Figure 7.)

With these definitions we have

$$t(L) = \text{number of splices to obtain } L \text{ from } K\,.$$

$$t_-(L) = \text{number of negative splices}\,.$$

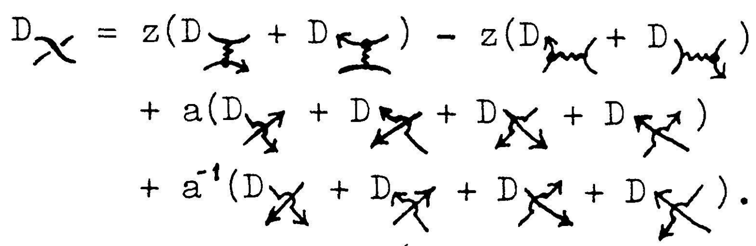
$\mathrm{Au}(K, T)$ = set of admissible unlinks relative to K and T (here K is unoriented).

Then the Dubrovnik polynomial is given by the formula

$$D_K = \sum_{L \in \mathrm{Au}(K,T)} (-1)^{t_-(L)} z^{t(L)} a^{w(L)} \mu^{|L|-1}$$

$$\left(\mu = 1 + (a - a^{-1})/z\right).$$

As a state-expansion we can write



Once again,

$$D_K = \sum_{L \in \mathrm{Au}(K,T)} \langle K \mid L \rangle\, \mu^{|L|-1}$$

where $<K \mid L>$ denotes the product of vertex weights (all relative to the given template). Independence of the template follows from the well-definedness of the polynomial itself.

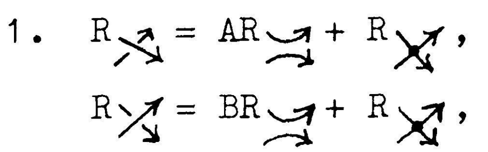
Remark. It would be very interesting to know the relationship between this state model for the Kauffman polynomial and the extraordinary model of Jaeger [34]. Jaeger gives a state expansion where the states are a collection of oriented knots and links. Each state is itself evaluated via the regular isotopy version of the Homfly polynomial.

V. Graph polynomials

The two skein polynomials (Homfly and Kauffman) each have three variable extensions to rigid vertex isotopy invariants of 4-valent graphs imbedded in three-space. This construction has been announced in [45]. (See also [56] and [74].) Our skein models involve 4-valent graphs implicitly, and so give rise to a natural definition for these extended polynomials as state models.

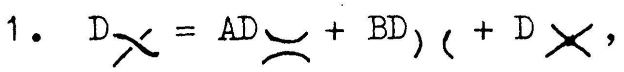
Let the new variables A and B be given, with $z = A - B$ the usual z for the skein polynomials. The extended polynomials are then defined by the axioms:

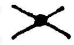
Homfly Extension Axioms



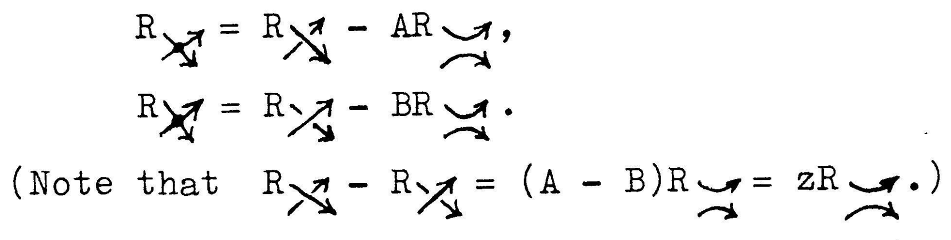
2. R_K = usual regular isotopy Homfly polynomial if K is free of graphical vertices ().

Kauffman Extension Axioms



2. D_K = usual regular isotopy Dubrovnik polynomial if K is free of graphical vertices ().

In each case, these axioms are taken as the *definition* of the polynomial in the case of the presence of graph vertices. That is, one rewrites



as the (recursive) definition in terms of an expansion involving the standard link polynomial (evaluated on a collection of links obtained by removing graph vertices and replacing by splices or by crossings). It is easy to verify that the resulting graph polynomial is well-defined *on the basis of the theory of the Homfly or Kauffman polynomial.*

By using the skein models we can give a direct formula for the evaluation of the extended polynomial on a planar graph. (See below.) This gives another point of view for these polynomials. In this view the axioms for the extended polynomials give state expansions for them whose states are plane graphs. Each plane graph state contributes a polynomial to the summation — weighted by A's and B's that tell how it was made planar by projecting and splicing.

In this view, all the complexity of the skein models for these polynomials has been absorbed into the extended polynomial evaluations for plane graphs.

Here are the explicit formulas for the Homfly case. I leave the case of the Kauffman extension to the reader. (See also [56].)

LEMMA 5.1. *Let G be a plane oriented 4-valent graph. Let $U(G)$ be the set of graphs obtained from G by splicing (oriented splice) a subset of vertices from G. Let T be a template for G (regard G as a universe and use the notion of template in section 2).*

For each H in $U(G)$ give H the structure of a standard unlink $L = L(H)$ relative to the template T. Let $t(H), t_+(H), t_-(H)$, denote the number of splices, positive splices, negative splices of $L(H)$. Let $w(H)$ denote the writhe of $L(H)$. Then

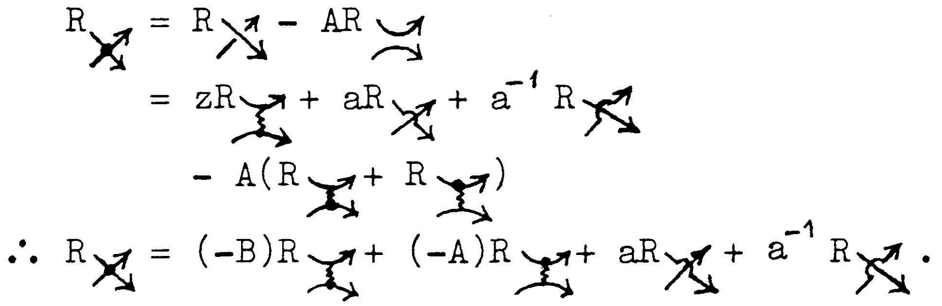
$$R_G = \sum_{H \in U(G)} (-1)^{t(H)} A^{t_-(H)} B^{t_+(H)} a^{w(H)} \delta^{|H|-1}$$

where

$$\delta = (a - a^{-1})/z = (a - a^{-1})/(A - B),$$

and $|H|$ denotes the number of crossing circuits in H (i.e. the number of link components in $L(H)$).

Proof. We give a state expansion for the extended polynomial by using the axiom. In abbreviated form this gives



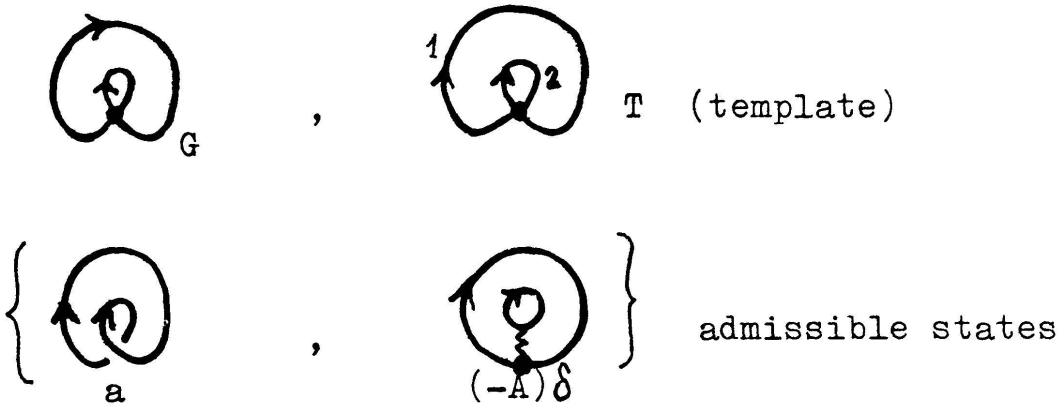
$(z = A - B)$.

Hence, the vertex weights for R_G are $(-B)$ for a positive splice, $(-A)$ for a negative splice, and otherwise the same as in the model for the Homfly. This completes the proof.

In the case of both D_G and R_G for plane graphs it would be good to verify their properties and definedness directly and independently of the knot theory. The knot theory of the (extended) Homfly and Kauffman polynomials would then be seen to rest on this theory of plane graph polynomials.

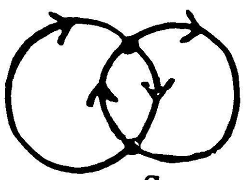
An example.

1.

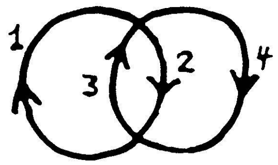


$$\therefore R_G = a - \frac{A(a-a^{-1})}{A-B} = \frac{Aa^{-1} - Ba}{A-B}$$

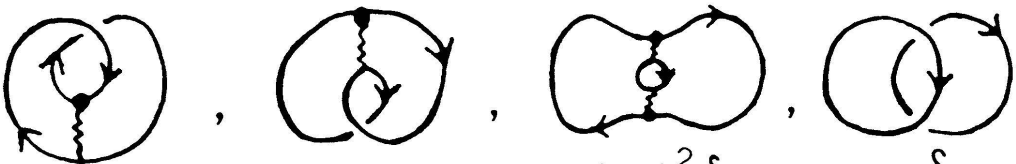
2.



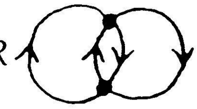
G



T (template)

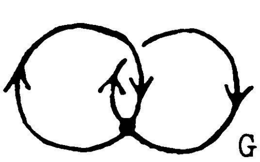


$\{$ $-Ba$, $-Aa$, $(-A)^2\delta$, δ $\}$

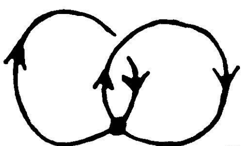
$\therefore R$  $= -Ba - Aa + (A^2+1)\dfrac{(a-a^{-1})}{A-B}$

$= [(a-a^{-1})+(B^2a-A^2a^{-1})]\,(A-B)\,.$

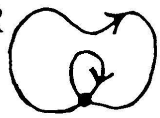
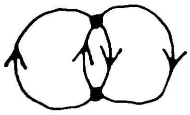
3.



G



G*

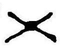
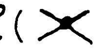
$R_G = AR$  $+ R$ 

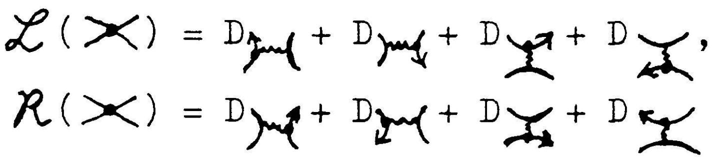
$$= A\left[\frac{Aa^{-1}-Ba}{A-B}\right] + \left[\frac{(a-a^{-1})+(B^2a-A^2a^{-1}}{A-B}\right]$$

$$= \frac{(a-a^{-1}) + (B^2-AB)a}{A-B}\,.$$

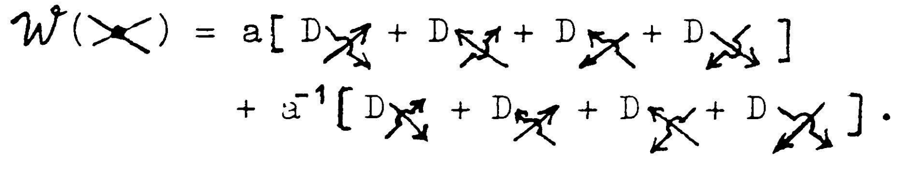
Here the rational function is not invariant under the (simultaneous) substitution of a by a^{-1} and interchange of A and B. This reflects the fact that the graph embeddings G and G^* are not rigid-vertex isotopic.

It is worth mentioning the planar graph polynomial in the Dubrovnik case. The result is

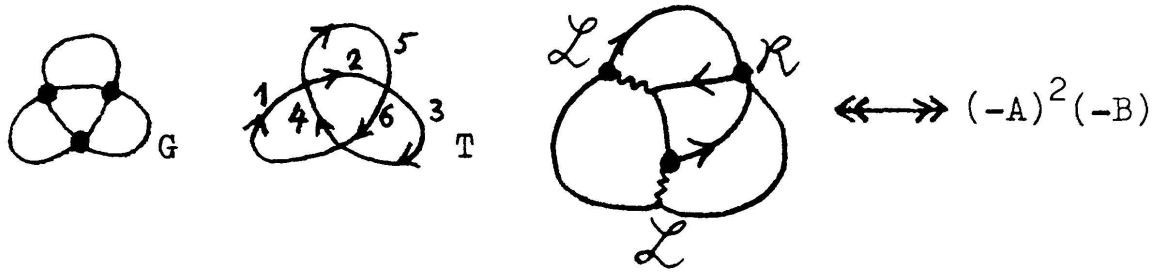
D  = (-A)$\mathcal{L}$() + (-B)$\mathcal{R}$() + $\mathcal{W}$(), where



and



A state in this expansion is obtained by first splitting (in any way) the vertices of the given unoriented four-valent plane graph G. The vertex weights are then determined by the template, as illustrated below.



If $<G \mid S>$ denotes the product of vertex weights for a given state S, then the polynomial has the form

$$D_G = \sum_S <G \mid S> \mu^{|S|-1}, \quad \mu = 1 + (a - a^{-1})/(A - B).$$

Proof of these formulas from the extension axioms follows just as in the Homfly case.

VI. The Conway Polynomial

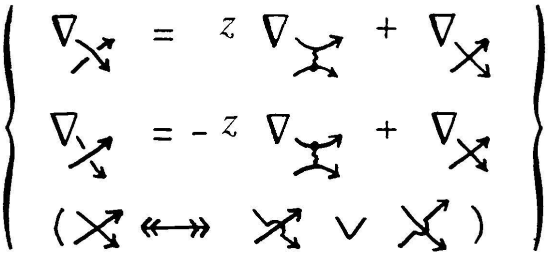
The skein models give a very elegant formulation of the Conway polynomial ([16], [41]) (compare [33])

$$\nabla_K(z) = R_K(z, 1).$$

Specializing the formula for the skein model we have

$$\nabla_K(z) = \sum (-1)^{t_-(L)} z^{t(L)}$$

(summation over $L \in A(K. T)$, $|L| = 1$),



(Notation of section 2.)

Note that each state in this model has a single crossing circuit. Hence the template can be replaced by a single choice of base-point.

Writing

$$\nabla_K(z) = a_0(K) + a_1(K)z + a_2(K)z^2 + \cdots,$$

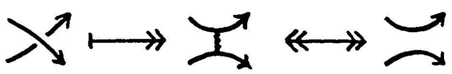
we have

$$a_n(K) = \sum(-1)^{t_-(L)}$$

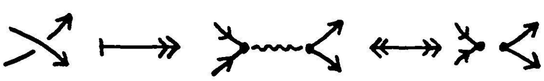
(summation over $L \in A(K, T)$, $|L| = 1$, $t(L) = n$).

Note that the second coefficient, $a_1(K)$, is the linking number of K when K is a 2-component link. The coefficients are generalized (self-)linking numbers.

It is worth comparing this model with the model for the Conway polynomial given in *Formal Knot Theory* [42]. I shall refer to the latter model as the FKT model. The FKT model sums over all *Jordan Euler Trails* on the universe underlying the link diagram K. These trails result from splicing the diagram K at each crossing in either oriented

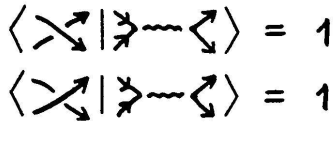


or non-oriented



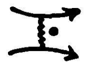
fashion. A choice of basepoint determines the vertex weights via the rules:

1. The unoriented splice has weight one.



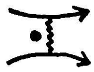
2. *Notation.*

Let



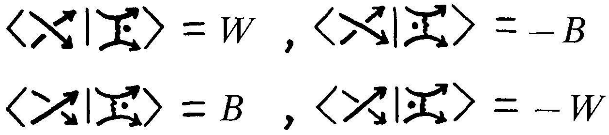
mean that the first passage through this site from the basepoint is *in the direction of the arrows.*

Let

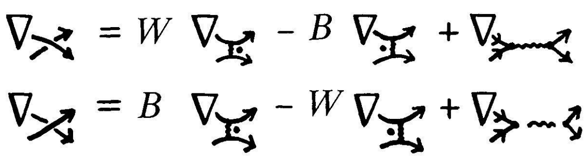


mean that the first passage through this site from the basepoint is *opposite to the direction of the arrows.*

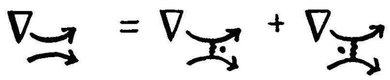
Then the weights are:



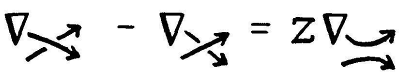
These vertex weights give the state expansion formulas:



with $z = W - B$, and $WB = 1$ (for topological invariance). Note that



is a tautology, and hence the Conway exchange identity



follows at once.

Definedness and properties of this model rest on a combinatorial result (the Clock Theorem [42]) from which it is straightforward to verify invariance under the Reidemeister moves. Furthermore, the model extends to a state model for the multi-variable Alexander-Conway polynomial (one variable for each component in a link). The FKT model is very closely related to Alexander's original approach to the polynomial via the Dehn presentation of the fundamental group of the link complement [6].

The FKT model has a number of intriguing features. It calculates a determinant of a generalized Alexander matrix. It is the low temperature limit of a generalized Potts model [57].

Is the FKT model a reformulation of the skein model for the Conway polynomial? There are a number of ways to try to generalize the FKT model to obtain a model of the Homfly polynomial. An answer to this question would shed light on the relationship of the FKT model and the Homfly polynomial. (And consequently on the relationship of the Homfly polynomial and the fundamental group of the link.)

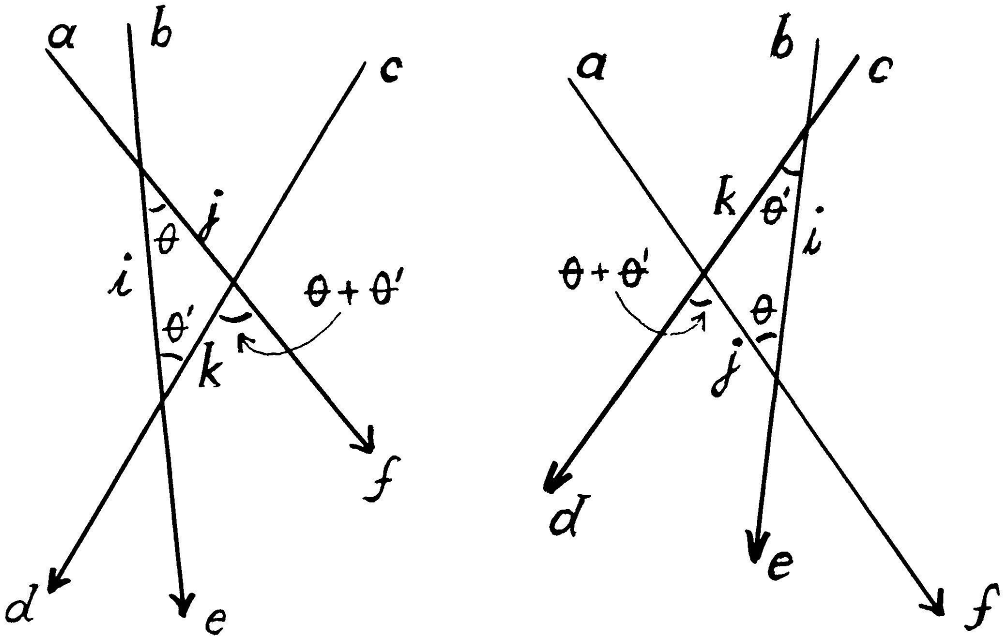
VII. Yang-Baxter Models

I now turn to state models for specializations of the Homfly and Kauffman polynomials that arise from solutions to the Yang-Baxter Equation [10]. These models were devised by Vaughan Jones (Homfly) ([40]) and Volodja Turaev (Kauffman) ([93]). (See also the series of papers ([1], [2], [3], [4], [5], [64]) by Akutsu, Wadati and collaborators.) The reformulation of these models as given here is due to the author (compare [55], [58]).

The Yang-Baxter Equation arises in the study of two-dimensional statistical mechanics models [10] and also in the study of 1 + 1 (1 space dimension, 1 time dimension) quantum field theory ([25], [100]). In the latter case, the motivation and relationship with knot theory is easiest to explain.

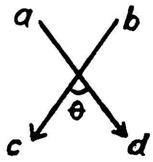
Regard a crossing in a universe (shadow of a link diagram) as a diagram for the interaction of two particles. Label the in-going and out-going lines of an oriented crossing with the "spins" of these particles. (Mathematically, spin is a generic term for a label chosen from an ordered index set $\mathscr{I}$. In applications it may denote the spin of a particle, or it may denote charge or some other intrinsic quantity.) The angle between the crossing segments can be regarded as an indicator of their relative momentum (rapidity). For each assignment of spins and each angle θ there will be a matrix element that, in the physical context, measures the amplitude (complex probability amplitude) for the process with these spins and rapidity.

The S matrix, $S^{ab}_{cd}(\theta)$, is said to be *factorized* if it satisfies the equations shown in Figure 8. This matrix equation is the Yang-Baxter Equation. Physically, it means that amplitudes for multi-particle interactions can be calculated from the two-particle scattering amplitude.



$$\sum_{i,j,k\in\mathscr{I}} S^{ab}_{ij}(\theta)\, S^{jc}_{kf}(\theta+\theta')\, S^{ik}_{de}(\theta') = \sum_{i,j,k\in\mathscr{I}} S^{bc}_{ki}(\theta')\, S^{ak}_{dj}(\theta+\theta')\, S^{ji}_{ef}(\theta)$$

Yang-Baxter Equation



$\longleftrightarrow \quad S^{ab}_{cd}(\theta)$

FIGURE 8

As is evident from Figure 8, the Yang-Baxter Equation expresses an invariance related to the type III (triangle) Reidemeister move in knot theory. Figure 9 illustrates the beginning of this correspondence. In this Figure 1 have matched the interaction picture for the S-matrix with a crossing of positive type, and have matched a related matrix, $\bar{S}(\theta)$, with a crossing of negative type. Here we assume that the matrix product

$$S^{ab}_{ij}(\theta)\, \bar{S}^{ij}_{cd}(-\theta) = \delta^a_c\, \delta^b_d$$
(sum on i and j)

is the identity matrix (indicated with Kronecker deltas above). In this form, both crossing and reversed crossing measure the same underlying momentum — corresponding to the (counterclockwise) measure of the angle between the crossing lines.

Switching the crossing corresponds to this step in inverting the S-matrix (that must be combined with a reverse momentum difference to actually obtain the inverse). In the case of a *special S-matrix* (see below and Figure 11) we will accomplish the momentum change with extra interactions (angles in the diagram) so that a crossing and its reverse can cancel.

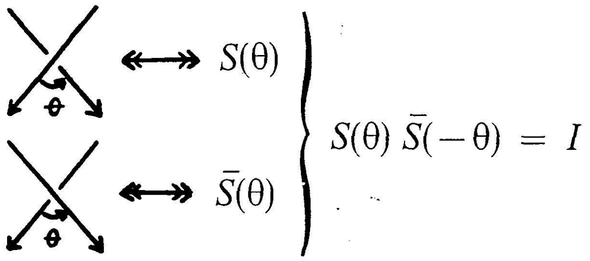
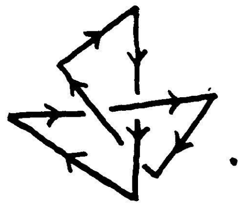


FIGURE 9

It is a curious and deep fact that there is this correspondence between a topological move for weaves in three-dimensional space and the factorizability condition for the S-matrix.

In fact, by assuming that the solution to the Yang-Baxter equation has a special form, one can produce state models for link invariants! In the first version discussed here I shall use piecewise linear (pl) link diagrams. In a pl diagram it will be assumes that each segment of a crossing forms a straight line at the crossing. Along with crossings there are isolated vertices



The angle at a vertex is measured as the angle between the incoming segment and the outgoing segment with direction as shown in Figure 10.

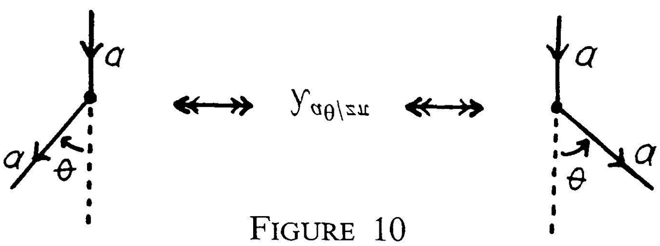


FIGURE 10

One may think of the isolated vertices as interactions with an external field that causes the trajectory of the particle to change direction.

It is assumed that *spin is preserved at all sites of interaction.* Thus, at a 4-vertex we have $a + b = c + d$, and at a 2-vertex we have that the incoming and outgoing spins are identical.

We shall assume that $S(\theta)$ has the following special form:

$$S^{ab}_{cd}(\theta) = R^{ab}_{cd}\,\lambda^{\theta(d-a)/2\pi}$$

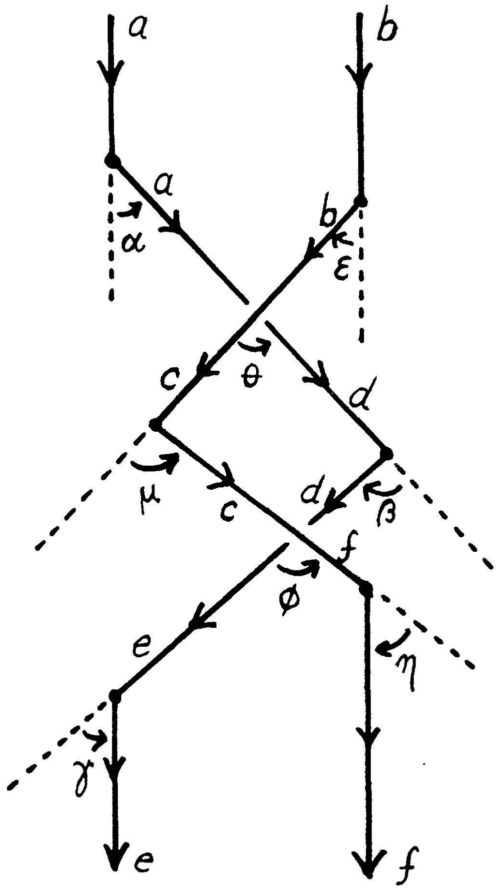
where λ is an (as yet) unspecified variable, and R is an invertible matrix, with no θ-dependence. We also assume that $a + b = c + d$ whenever $R^{ab}_{cd} \neq 0$. Call such an S a *special S-matrix.*

Note that $R = S(0)$ satisfies the Yang-Baxter equation without the angular parameter. Let $\bar{R}$ denote the inverse matrix to R, and let $\bar{S}$ be defined so that $\bar{S}(0) = \bar{R}$:

$$\bar{S}^{ab}_{cd}(\theta) = \bar{R}^{ab}_{cd}\,\lambda^{\theta(d-a)/2\pi}.$$

Remark. A little Euclidean Geometry shows that the oriented type II move works perfectly with respect to this definition when $\bar{S}$ is associated with a crossing of negative type. See Figure 11.

Define the following *vertex weights* for piecewise-linear link diagrams as indicated in Figure 12. In this Figure the local state configuration is shown in the right-hand of the bracket, and the crossing type in the left-hand half.



$\alpha + \beta + \gamma = 0$
$\varepsilon + \mu + \eta = 0$
$\theta + \phi = \mu - \beta$

$a + b = c + d$
$c + d = e + f$
(spin conservation)

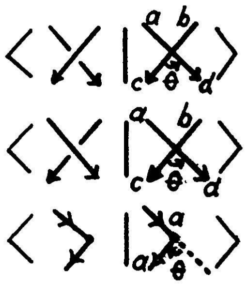
$\Rightarrow a\alpha + b\varepsilon + (d-a)\theta + \mu c + \beta d + (f-c)\phi + \gamma e + \eta f = 0$

$\Rightarrow$ product of vertex weights for diagram on left is $R^{ab}_{cd}\,\bar{R}^{cd}_{ef}$

$\Rightarrow$ invariance under pl type II move (by $R\bar{R} = I$).

Geometry of Oriented II — Move

FIGURE 11



$= R^{ab}_{cd}\,\lambda^{\theta(d-a)/2\pi}$

$= \bar{R}^{ab}_{cd}\,\lambda^{\theta(d-a)/2\pi}$

$= \lambda^{a\theta/2\pi}$

Vertex Weights

FIGURE 12

For this Yang-Baxter model, a *state* of the oriented universe U underlying a given link diagram K is *any assignment of spins (from the index set* $\mathcal{I}$*) to the edges of* U (modulo conservation of spin).

Given a diagram K and a state σ, let $<K \mid \sigma>$ denote the *product* of these vertex weights. Let $<K>$ denote the sum of such products taken over all (spin-conserving) states.

LEMMA 7.1. *Let* $S(\theta)$ *be a special factorized S-matrix. Let the state summation*

$$<K> = \sum_{\sigma} <K \mid \sigma>$$

be defined on oriented link diagrams K *as explained above. Then, if* $R = S(0)$ *satisfies the additional condition*

$$(*) \qquad \sum_{i,\, j \in \mathcal{I}} R^{ja}_{ic}\, \bar{R}^{ie}_{jf}\, \lambda^{\frac{c-j}{2}}\, \lambda^{\frac{f-i}{2}} = \delta^{a}_{f}\, \delta^{e}_{c}$$

$$\Leftrightarrow\ S(-\pi)\,\bar{S}(\pi) = I \quad \text{where}$$

$$S^{ab}_{cd} = S^{ca}_{db}, \qquad S^{ab}_{cd} = S^{bd}_{ac}$$

$\Leftrightarrow$ Cross Channel Inversion

$\Leftrightarrow$ Inversion under II-move with reverse orientation

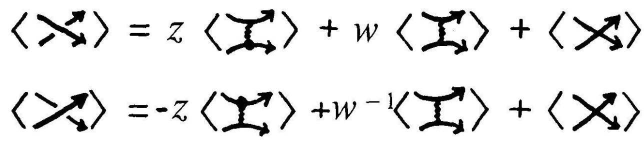
then $<K>$ *is an invariant of regular isotopy.*

(See [40] or [58] for a proof of this lemma.)

Remark. There is a corresponding state model and lemma for unoriented links.

The lemma shows that any angle-free, invertible solution R of the Yang-Baxter Equation gives rise to a regular isotopy invariant of knots and links (by choosing λ to satisfy (*)).

We now exhibit two such solutions in a form that emphasizes their formal similarity to the skein models. In these models the angle (rapidity) terms can all be relegated to counting the number of circuits in a state (with multiplicity). In the Homfly case the model takes the form



$(z = w - w^{-1})$

Note the similarity of the formalism with that for the skein model. Here however, I adopt the convention that *the dotted segment has a smaller spin* than the un-dotted segment. The local state without dots has equal spins on its lines. Spins must be different for crossing lines. The vertex weights of the expansion correspond to a particular solution of the Yang-Baxter Equation.

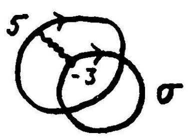
In this model a state σ is a splitting of the universe (i.e. splice a subset of its crossings) and a labelling of the circuits by spins. (The circuits are not allowed to cross themselves.) The value of a state is $\lambda^{\|\sigma\|}$

where
$$\|\sigma\| = \sum_{\substack{\text{circuits} \\ C \text{ in } \sigma}} \text{label}(C) \cdot \text{rot}(C)$$

with

$$\text{rot}(⟲) = +1, \quad \text{rot}(⟳) = -1.$$

e.g.



$$\sigma \Rightarrow \|\sigma\| = 5(-1) - 3(-1) = -2.$$

and label(C) is the spin assigned to the (edges of) the circuit C. In this model the state value $\lambda^{\|\sigma\|}$ is a summary of all the angle contributions in the pl formulation. The weights from the set $W = \{z, -z, w, w^{-1}, 0, 1\}$ are the values taken by the angle-independent part R of the S-matrix. This model is expressed for arbitrary link diagrams in the form

$$<K> = \sum_{\sigma} <K \mid \sigma> \lambda^{\|\sigma\|}$$

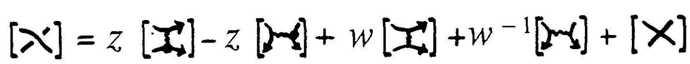
where $<K \mid \sigma>$ denotes the product of vertex weights from the set W arising from the expansion given above.

A particularly nice model occurs for index set in the form

$$\{-n, -n+2, \ldots, n-2, n\} \quad \text{with} \quad \lambda = w\,.$$

This gives a series of one-variable specializations of the Homfly polynomial. (See [40], [58], [93].) *Is there a Yang-Baxter model for the full Homfly polynomial?* This is an open question.

A similar approach works for the Dubrovnik form of the Kauffman polynomial. See [58], [93]. The expansion formula has the appearance.



(It is understood that reversing the orientation of a line is accompanied by the negation of its spin.) Once again, the dot on a line means that it has smaller spin.

VIII. Applications and questions

This section is devoted to a few applications of the skein and state models and related questions.

1. Let ∇_K denote the Conway polynomial. The skein model is embodied in the formula of section 6:

$$\nabla_K = \sum_{L, |L| = 1} (-1)^{t^-(L)} z^{t(L)}$$

from which we see easily that

$$\max \deg \nabla_K \leqslant V - S + 1 = \rho(K)$$

where V is the number of crossings in the diagram K, S is the number of Seifert circuits (the set of circuits obtained by splicing all crossings of K). On knows that $\rho(K) = \operatorname{rank}(H_1(F))$ where F is the Seifert spanning surface [42] corresponding to the diagram K. If K is an alternating link then $\max \deg \nabla_K = \rho(K)$ [76]. This is generalized to the class of alternative links in [42], using the FKT model. Is there a proof using the skein model? [1])

In the case where all the crossings are of positive type, we see from the skein model that all terms of ∇_K are positive, and it is then easy to see that the highest degree term is of degree $\rho(K)$.

[1]) *Note added in proof:* A proof using the skein model for the theorem on alternative links has been found by John Mathias — University of Maryland, May 1989.

2. Similar remarks apply to the Homfly model of section 3. In the case of the Yang-Baxter model for the Homfly polynomial given in section 7, it is easy to see that the highest z-degree is $\rho(K)$ when K is positive — this time by constructing an appropriate spin state.

3. Thistlethwaite [89] proves that the writhe $w(K)$ is an ambient isotopy invariant for K alternating and reduced. It would be useful to see a proof of this result using the skein model for D_K (section 4).

4. The Alexander polynomial Δ_K is given by the formula

$$\begin{aligned}\Delta_K(t) &\doteq \nabla_K(\sqrt{t}-1/\sqrt{t}) \\ &= \sum_{|L|=1} (-1)^{t-(L)}(\sqrt{t}-1/\sqrt{t})^{t(L)}\end{aligned}$$

where $\doteq$ denotes equality up to sign and powers of t. One knows ([23]) that if K bounds a smooth disk in the upper 4-space $((x, y, z, t)$ with $t>\theta)$ then

$$\Delta_K(t) \doteq f(t)f(t^{-1})$$

for some polynomial $f(t)$. Can this fact be deduced directly from the skein model or from the FKT model? A solution should generalize to give new information about the full skein polynomial behaviours on slice links.

IX. Relations with mathematical physics

I have deliberately included a description of the Yang-Baxter models in this paper in order to raise the question of the relation of the skein models to mathematical physics. The Yang-Baxter models can be regarded as averages of scattering amplitudes over all possible spin states — hence as discrete Feynman integrals, or as partition functions for two-dimensional statistical mechanics models. The FKT model for the Conway polynomial can be seen [57] as the low temperature limit of a partition function of a generalized Potts model.

Meta-time

If we interpret the FKT model or the skein models in a particle interaction framework, then a curious and interesting issue arises:

Think of a particle moving forward and backward in "time" on a given universe. The "same" particle may traverse a given site (crossing) twice.

Locally this appears as an interaction of two distinct particles. But in the mathematical trajectory one of these particles came through the site "first" (all dependent on the basepoint or template). And the "way" the particle comes through first may make it particle or anti-particle on this "first pass".

In the skein and FKT models the local vertex weights depend upon this aspect of global trajectory structure. The skein models care which strand is the locus of the "first pass" and the FKT model needs to know whether the first pass is a particle (with the local time-line) or an anti-particle (against the local time-line). Remarkably and fortunately for knot theory, the total summation over all trajectories (denoted $<K>$ in both models) is independent of the choice of template. Thus the first objection to considering "meta-time discriminations" disappears in the averaging. These models solve an issue of invariance that must arise for scattering amplitudes that include the issue of meta-time ordering. It would be very useful to see a parallel problem on this theme in mathematical physics proper.

Into the Third Dimension

Find definitions of the invariants that are intrinsically three-dimensional — that do not depend upon the use of a diagram. This is solved classically for the Alexander polynomial (see [12], [14], [43], [82] for various accounts). The problem remains open in its full generality for the other skein polynomials.

Witten [99] has suggested a definition as the integral over all gauge connections of the trace of the holonomy of the connection — along the knot. The measure in the integral is weighted by the Chern-Simons Lagrangian. Witten's work may well be the desired answer to the question of an intrinsic definition. If so, then there arises a host of questions about the relationship of the holonomy and state models approaches. (Compare [71], [83], [96].)

In the case of the skein models, the relationship is direct. Once the exchange relation is proved from properties of the holonomy, the skein model follows just as we have constructed it here. The question about "meta-time" is then transposed to the more concrete matter of keeping track of orders of computation of holonomies on circuits in the diagram.

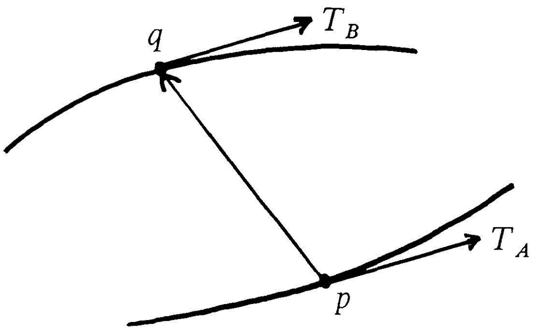
More generally, the possibility of intrinsic models raises the question of the *nature of a crossing in a link diagram.* From the point of view of the plane, a crossing is where the plane curve interacts with itself or with another component. From the point of view of space, the crossing is two distinct points in the space-curve that are projected to the same point in the plane for a given direction of projection.

The relation of these two points of view is seen in sharp relief in the Gauss definition of the linking number of two curves A and B in three-space:

$$\mathrm{lk}(A, B) = \iint_{A \times B} \mathcal{G} ds_A ds_B$$

Here the Gauss Kernel $\mathcal{G}$ is given by the formula

$$\mathcal{G} = e \cdot (T_A \times T_B)/\| e \|^3$$



where e is a direction vector between two points, p and q, one from each curve. T_A and T_B are unit tangent vectors to the curves at these points.

Note that the Gauss kernel vanishes whenever the direction vector and the two tangent vectors occupy the same plane. Thus if we use the Gauss definition to calculate the linking number for a nearly planar diagram, then the result is a sum of vertex contributions (the neighborhoods of the vertices are the only contributors to the integral). (See [27], [97].)

It appears that the Gauss kernel holds a clue to how the state models (also built from local vertex contributions) can be defined three-dimensionally. In such a model, states would be defined directly on the space curve, and vertex weights would be replaced by weights of self-interaction of the knot, modulated by the Gauss kernel.

To see the approximate form of such a model, suppose that we are generalizing the Yang-Baxter model for pl diagrams. Rewrite the model so that it is in exponential form

$$<K> = \sum_{\sigma} e^{[K|\sigma]}$$

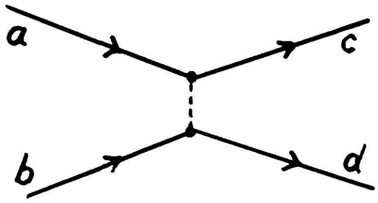
Then sums of vertex weights appear in the exponents. These divide into weights from 2-vertices (corresponding to curvature) and weights from interactions. The interaction weights depend upon angle and local spins — all information that is available on projecting in a given direction e.

This suggests the form of a model as

$$<K> = \int_{\sigma} d\sigma \exp\left(\iint_{K\times K} [p, q \mid \sigma]\,\mathscr{G} + \int_{K} [p \mid \sigma]\right)$$

where the problem of taking the limit over subdivisions of the space curve, and the definition of the limiting state space is certainly unsolved.

The proposed model is designed as a generalization of the planar models. The crossings are replaced by pairs of points on the curve, and the Gauss kernel appears, as in the linking number. In a piecewise-linear approximation to the model, the space curve is divided into straight-line segments. A state assigns a spin to each segment. If spin remains unchanged at a vertex, then the vertex contributes a simple angular term, as in the planar case. *If spin changes at a vertex, then this vertex must be paired up with another vertex so that the pair can be regarded as under-going a spin preserving interaction.*



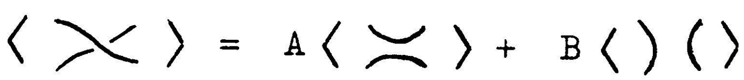
$a + b = c + d$

This is the generalization of the crossing in the planar case. A state is admissible if it is configured with such self-interactions allowing spin conservation. The three-dimensional approximation sums over all such admissible states.

Appendix on state model formalism

This appendix is a short note on the formalism I use for expressing state models.

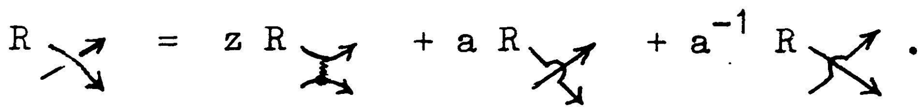
The bracket polynomial [41] is defined via equations of the form



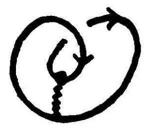
$$\langle OK \rangle = d \langle K \rangle$$

(It is a regular isotopy invariant when $B = (1/A)$ and $-d = A^2 + A^{-2}$.) Here the small diagrams stand for parts of larger diagrams that differ only as shown in the small diagrams. Since all diagrams are of the same type (unoriented link diagrams) these equations are easily understood.

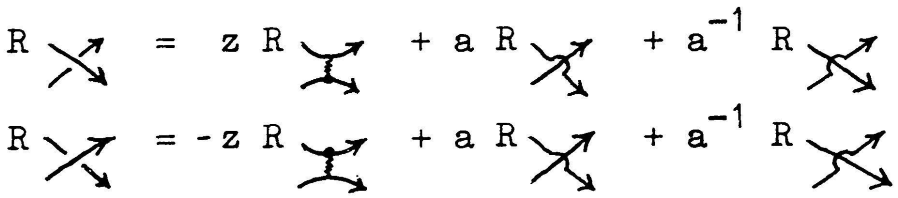
On the other hand we have used formalisms in this paper of the ilk



Here the small diagrams on the right-hand side of the equation replace a knot-diagrammatic crossing. The large diagrams on the right go out of the category of link diagrams to an appropriate category of labelled graphs. As a result (for example), one may legitimately ask: What is the value of the R-polynomial on the "mixture" shown below?

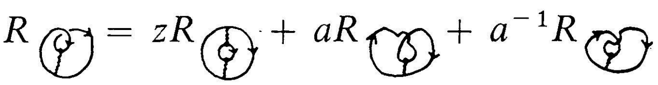


To answer this type of question in all cases I take the following point of view: *Equations such as*



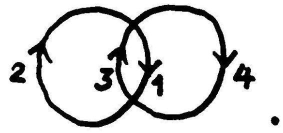
are a recursive algorithm for expressing R *as a sum of evaluations of decorated graphs – all of whose knot-theoretic crossings have been replaced by one of the choices in these equations.* A further rule must be given to specify the values of the decorated graphs (the "states") produced by this algorithm. (In the case of the R-polynomial this rule involves using the template (Section 2) to determine if the graph is admissible. If so, then its value is $\delta^{|S|-1}$ where $|S|$ is the number of circuits in the state S.)

Thus

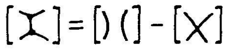


$$= z\delta + a^{-1}$$

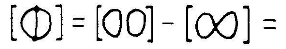
for the template



There are many instances of this sort of expansion outside of the theory of knots and links. For example, the following expansion (compare [78]) for *trivalent plane graphs G*



gives states that are locally four-valent plane graphs. If the value of a state S is taken to be *three raised to the number of crossing circuits in* S, then $[G]$ is the number of colorings of the edges of G with three colors so that three distinct colors meet at each vertex of G. The existence of such a coloring for a trivalent plane graph is well known to be equivalent to finding a four-coloring of its faces so that no two faces that share an edge receive the same color. It is a delicate matter to determine when $[G]$ is non-zero.

 $= 3^2 - 3 = 6$.

Other conventions, more closely related to tensor formalisms are discussed in [78] and [58].

In general, these pictorial expansions are a way to express the vertex weights of a model in a fashion that is easy to relate with the geometry of the diagrams themselves.

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